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Data-driven Set-based Estimation of Polynomial Systems with Application to SIR Epidemics

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***Abstract*— This paper proposes a data-driven set-based estimation algorithm for a class of nonlinear systems with polynomial nonlinearities. Using the system's input-output data, the proposed method computes a set that guarantees the inclusion of the system's state in real-time. Although the system is assumed to be a polynomial type, the exact polynomial functions, and their coefficients are assumed to be unknown. To this end, the estimator relies on offline and online phases. The offline phase utilizes past input-output data to estimate a set of possible coefficients of the polynomial system. Then, using this estimated set of coefficients and the side information about the system, the online phase provides a set estimate of the state. Finally, the proposed methodology is evaluated through its application on SIR (Susceptible, Infected, Recovered) epidemic model.**

I. Introduction

For monitoring and control of dynamical systems, the knowledge of the state is undeniably crucial. Typically, observer design techniques rely on the system's model for estimating the state in real-time. However, as modern engineering systems are becoming increasingly complex, developing accurate models to describe a system's behavior is quite challenging [1], [2]. This motivates the development of a data-driven approach for set-based state estimation.

Given bounded uncertainties and measurement noise, estimating a set that guarantees the inclusion of the true system state at each time step is a classical problem studied in [3]. Several set-based estimation approaches have been presented in the literature. The interval observers [4]–[6] utilize two observers to estimate the upper and lower bounds of the state trajectory and rely on the monotonicity properties of the estimation error dynamics to ensure that the true state remains inside the estimated bounds. Secondly, the set-membership observers [7]–[9] intersect the state-space regions consistent with the model with those obtained from the online measurements to estimate the current state set. Finally, zonotopic filtering [10]–[13] provides set-based state estimation using interval arithmetic and zonotopes under bounded uncertainties in the model. See [14], [15] for a comprehensive review of the existing set-based estimators.

The limitation of the existing methods for set-based estimation is their assumption of an a priori known model. Such a model is not available in many applications or too costly to develop and identify. Furthermore, relying on inaccurate models may violate the formal guarantees of the system. The data-driven paradigm is therefore gaining precedence over the model-based paradigm because of the possibility to obtain huge amounts of data from the system thanks to the advancements in sensor technologies. Recently, several studies are dedicated to data-driven reachability analysis [16]–[24] that overcome the limitation of prior model knowledge. However, to the best of our knowledge, only one work presented a set-based estimation technique for linear systems [25] given the offline data and online measurements.

In this paper, we propose a set-based estimator for nonlinear polynomial systems by extending [25]. We provide formal guarantees for the proposed data-driven set-based estimation method and show its effectiveness on the application of compartmental SIR (Susceptible, Infected, Recovered) epidemic process. The set-based estimation for epidemics is essential as it provides formal guarantees on the bounds of the infected population in the presence of uncertainties and discrepancies in the data. The existing set-based estimators [26]–[28] for compartmental epidemics rely on a model and a priori knowledge of some of its parameters. In contrast, our method does not assume such an a priori knowledge. However, if the system's model structure is known, our method can incorporate it as a side information to further refine the set-based estimation results.

II. Preliminaries and Problem Statement

A. Notations and Set Representations

The set of real and natural numbers are denoted as $\mathbb{R}$ and $\mathbb{N}$, respectively, and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. The transpose and Moore-Penrose pseudoinverse of a matrix X are denoted as X^T and $X^\dagger$, respectively. If $X \in \mathbb{R}^{n\times m}$ is full-row rank, then $X^\dagger = X^\mathsf{T}(XX^\mathsf{T})^{-1}$ is the right inverse, i.e., $XX^\dagger = I_n$, where $I_n \in \mathbb{R}^{n\times n}$ is the identity matrix. The standard unit vector e_i is the i-th column of I_n, for an appropriate n, and $E_i = \text{diag}(e_i)$. The i-th element of a vector or list A is denoted by $A^{(i)}$.

Definition 1. (Zonotope [29]) Given a center $c \in \mathbb{R}^n$ and $\xi \in \mathbb{N}$ generator vectors $g^{(1)}, \dots, g^{(\xi)} \in \mathbb{R}^n$, a zonotope is the set

$$\mathcal{Z} = \Big\{ x \in \mathbb{R}^n \,\Big|\, x = c + \sum_{i=1}^{\xi} \beta_i \, g^{(i)} \, , -1 \le \beta_i \le 1 \Big\}$$

denoted as $\mathcal{Z} = \langle c, G \rangle$ with $G = [\, g^{(1)} \, \dots \, g^{(\xi)} \,]$.

A linear map $L : \mathbb{R}^n \to \mathbb{R}^m$ applied to a zonotope $\mathcal{Z}$ yields $L\mathcal{Z} = \langle Lc, LG \rangle$. Given two zonotopes $\mathcal{Z}_1 = \langle c_1, G_1 \rangle$

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and $\mathcal{Z}_2 = \langle c_2, G_2 \rangle$ that are subsets of $\mathbb{R}^n$, their Minkowski sum is given by $\mathcal{Z}_1 + \mathcal{Z}_2 = \langle c_1 + c_2, [\, G_1 \ G_2 \,] \rangle$. Note that $\mathcal{Z}_1 - \mathcal{Z}_2$ means $\mathcal{Z}_1 + (-1\mathcal{Z}_2)$.

Definition 2. (Constrained zonotope [30]) Given a center $\overline{c} \in \mathbb{R}^n$ and $\xi \in \mathbb{N}$ generator vectors $\overline{g}^{(1)}, \dots, \overline{g}^{(\xi)} \in \mathbb{R}^n$, and constraints $A \in \mathbb{R}^{n_c \times \xi}$ and $b \in \mathbb{R}^{n_c}$, a constrained zonotope is the set

$$\mathcal{C}=\Big\{ x \in \mathbb{R}^n \;\Big|\; x = \overline{c} + \sum_{i=1}^{\xi} \beta_i \, \overline{g}^{(i)},\; A\beta = b,\; -1 \leq \beta_i \leq 1 \Big\}$$

denoted as $\mathcal{C} = \langle \overline{c}, \overline{G}, A, b \rangle$ with $\overline{G} = [\, \overline{g}^{(1)} \ \dots \ \overline{g}^{(\xi)} \,]$.

Definition 3. (Matrix zonotope [31, p.52]) Given a center $C \in \mathbb{R}^{n \times k}$ and $\xi \in \mathbb{N}$ generator matrices $G^{(1)}, \dots, G^{(\xi)} \in \mathbb{R}^{n \times k}$, a matrix zonotope is the set

$$\mathcal{M} = \Big\{ X \in \mathbb{R}^{n \times k} \;\Big|\; X = C + \sum_{i=1}^{\xi} \beta_i \, G^{(i)}, -1 \leq \beta_i \leq 1 \Big\}$$

denoted as $\mathcal{M} = \langle C, \mathcal{G} \rangle$ with $\mathcal{G} = \{G^{(1)}, \dots, G^{(\xi)}\}$.

A linear map $L : \mathbb{R}^n \to \mathbb{R}^m$ applied to a matrix zonotope $\mathcal{M} = \langle C, \mathcal{G} \rangle$ yields $L\mathcal{M} = \langle LC, L\mathcal{G} \rangle$. Given two matrix zonotopes $\mathcal{M}_1 = \langle C_1, \mathcal{G}_1 \rangle$ and $\mathcal{M}_2 = \langle C_2, \mathcal{G}_2 \rangle$ that are subsets of $\mathbb{R}^{n \times k}$, their Minkowski sum is given by $\mathcal{M}_1 + \mathcal{M}_2 = \langle C_1 + C_2, \{\mathcal{G}_1, \mathcal{G}_2\} \rangle$.

Definition 4. (Constrained matrix zonotope [32]) Given a center $\overline{C} \in \mathbb{R}^{n \times k}$, generator matrices $\overline{G}^{(1)}, \dots, \overline{G}^{(\xi)} \in \mathbb{R}^{n \times k}$, and constraints $A^{(1)}, \dots, A^{(\xi)} \in \mathbb{R}^{n_c \times n_a}$ and $B \in \mathbb{R}^{n_c \times n_a}$, a constrained matrix zonotope is the set

$$\mathcal{N} = \Big\{ X \in \mathbb{R}^{n \times p} \;\Big|\; X = \overline{C} + \textstyle\sum_{i=1}^{\xi} \beta_i \, \overline{G}^{(i)}, \text{ where } \textstyle\sum_{i=1}^{\xi} \beta_i A^{(i)} = B \,, -1 \leq \beta_i \leq 1 \Big\}$$

denoted as $\mathcal{N} = \langle \overline{C}, \overline{\mathcal{G}}, \mathcal{A}, B \rangle$ with $\overline{\mathcal{G}} = \{\overline{G}^{(1)}, \dots, \overline{G}^{(\xi)}\}$ and $\mathcal{A} = \{A^{(1)}, \dots, A^{(\xi)}\}$.

Definition 5. (Interval matrix [31, p. 42]) An interval matrix $\mathcal{I} = \big[\, \underline{I}, \overline{I} \,\big]$ has intervals as its entries, where the left and right limits $\underline{I}, \overline{I} \in \mathbb{R}^{n \times k}$ are such that $\underline{I} \leq \overline{I}$ element-wise.

To over-approximate a zonotope $\mathcal{Z} = \langle c, [g^{(1)} \dots g^{(\xi)}] \rangle$ by an interval $\mathcal{I} = [\underline{i}, \overline{i}]$, we write $\mathcal{I} = \text{int}(\mathcal{Z})$ that is computed as

$$\overline{i} = c + \sum_{i=1}^{\xi} |g^{(i)}|, \quad \underline{i} = c - \sum_{i=1}^{\xi} |g^{(i)}|. \tag{1}$$

We compute the inverse of an interval matrix by following [33, Theorem 2.40], but other types of inverses provided in [34] can also be used. The pseudoinverse of an interval matrix $\mathcal{I}$ will also be denoted as $\mathcal{I}^{\dagger}$. We denote the interval vector (column) i of an interval matrix $\mathcal{I}$ by $\mathcal{I}(:, i)$.

B. Polynomial Systems

Consider a discrete-time nonlinear system

$$x(k+1) = f_p(x(k), u(k)) + w(k) \tag{2a}$$

$$y(k) = Hx(k) + v(k) \tag{2b}$$

where $f_p : \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \to \mathbb{R}^{n_x}$ is a polynomial nonlinearity with $p \in \mathbb{R}^{n_p}$ the vector of parameters, $w(k) \in \mathcal{Z}_w \subset \mathbb{R}^{n_x}$ is the process noise bounded by the zonotope $\mathcal{Z}_w$, $u(k) \in \mathbb{R}^{n_u}$ is the known input, $y(k) \in \mathbb{R}^{n_y}$ is the output measured by sensors with $n_y \leq n_x$, and $v(k) \in \mathcal{Z}_v \subset \mathbb{R}^{n_y}$ is the measurement noise of sensors bounded by the zonotope $\mathcal{Z}_v$. Without loss of generality, we assume that the output matrix $H \in \mathbb{R}^{n_y \times n_x}$ is full-row rank, i.e., rank$(H) = n_y$. Moreover, the initial state $x(0) \in \mathcal{X}_0$, for some known $\mathcal{X}_0 \subset \mathbb{R}^{n_x}$, and the system (2) is assumed to satisfy the observability rank condition in the sense of [35], [36].

In the interest of clarity, we will sometimes omit k as the argument of signal variables, however, the dependence on k should be understood implicitly. Let $n = n_x + n_u$ and $\zeta(k) = [\, x(k)^{\mathsf{T}} \ u(k)^{\mathsf{T}} \,]^{\mathsf{T}} = [\, \zeta_1(k) \ \dots \ \zeta_n(k) \,]^{\mathsf{T}} \in \mathbb{R}^n$. By a polynomial system, we mean that $f_p(\zeta) \in \mathbb{R}[\zeta]^{n_x}$ is a polynomial nonlinearity, where $\mathbb{R}[\zeta]^{n_x}$ is an n_x-dimensional vector with entries in $\mathbb{R}[\zeta]$, which is the set of all polynomials in the variables $\zeta_1, \dots, \zeta_n$ of some degree $d > 0$ given by

$$f_p^{(i)}(\zeta) = \sum_{j=1}^{m} \theta_j \zeta_1^{\alpha_{j,1}} \zeta_2^{\alpha_{j,2}} \dots \zeta_n^{\alpha_{j,n}}$$

with m the number of terms in $f_p^{(i)}(\zeta)$, $\theta_j \in \mathbb{R}$ the coefficients, and $\alpha_j = [\, \alpha_{j,1} \ \dots \ \alpha_{j,n} \,]^{\mathsf{T}} \in \mathbb{N}_0^n$ the vectors of exponents with $\sum_{i=1}^{n} \alpha_{j,i} \leq d$, for every $j \in \{1, \dots, m\}$.

C. Problem Statement

Given the input vector $u(k)$, output vector $y(k)$, output matrix H, and noise zonotopes $\mathcal{Z}_w$ and $\mathcal{Z}_v$, our main goal is to obtain a set-based estimate $\hat{\mathcal{R}}_k$ that guarantees the inclusion of the true state, i.e., $x(k) \in \hat{\mathcal{R}}_k$ given that n_x is known. Also, we aim to estimate the set of possible coefficients of the polynomial function $f_p(x(k), u(k))$.

III. Set-based Estimation Algorithm

We can write $f_p(\zeta)$ as follows (see [32] and [37])

$$f_p(\zeta) = \Theta_p \, h(\zeta) \tag{3}$$

where $h(\zeta) \in \mathbb{R}[\zeta]^{m_a}$ contains at least all the monomials present in $f_p(\zeta)$. These monomials can be included in $h(\zeta)$ if, for instance, the upper bound on the degree of polynomials in $f_p(\zeta)$ is known. Moreover, if the polynomial function $f_p(\zeta)$ is known, then $h(\zeta)$ contains all the monomials of $f_p(\zeta)$. The matrix $\Theta_p \in \mathbb{R}^{n_x \times m_a}$ contains the unknown coefficients of the monomials in $h(\zeta)$.

The proposed set-based estimator consists of two phases: offline and online, which are detailed below.

A. Offline Phase

In this subsection, we consider an offline phase, where it is assumed that an experiment is conducted and the data on the input $u(k)$ and the output $z(k)$ trajectories is collected for $k = 0, 1, \dots, T$, where

$$z(k) = Hx(k) + \gamma(k) \tag{4}$$

with $\gamma(k) \in \mathcal{Z}_\gamma \subset \mathbb{R}^{n_y}$ representing the noise bounded by the zonotope $\mathcal{Z}_\gamma = \langle c_\gamma, G_\gamma \rangle$. Notice that $z(k) \in \mathbb{R}^{n_y}$ denotes the data collected offline and, to avoid confusion, is distinguished from $y(k) \in \mathbb{R}^{n_y}$, which denotes the online sensor measurements in the next subsection. Given the above experiment, we obtain a sequence of noisy data and construct the following matrices of length T

$$Z^+ = \begin{bmatrix} z(1) & \dots & z(T) \end{bmatrix}, \; Z^- = \begin{bmatrix} z(0) & \dots & z(T-1) \end{bmatrix}$$
$$U^- = \begin{bmatrix} u(0) & \dots & u(T-1) \end{bmatrix} \tag{5}$$

and let $Z = \begin{bmatrix} Z^- & z(T) \end{bmatrix}$.

Assumption 1. It is assumed that $\|x(k)\|_\infty \leq \kappa$, for every $k \in \{0, \dots, T\}$ and some known $\kappa > 0$.

We aim to determine the mapping of the observation Z^+ and Z^- to the corresponding state-space region. In other words, we construct a zonotope $\mathcal{Z}_{x|z(k)} \subset \mathbb{R}^n$ that contains all *possible* $x(k) \in \mathbb{R}^{n_x}$ given the measurement $z(k) \in \mathbb{R}^{n_y}$, output matrix $H \in \mathbb{R}^{n_y \times n_x}$ and bounded noise $\gamma(k) \in \mathcal{Z}_\gamma$ satisfying (4). Precisely, we construct the set

$$\mathcal{Z}_{x|z(k)} = \left\{ x(k) \in \mathbb{R}^n \,\middle|\, Hx(k) = z(k) - \mathcal{Z}_\gamma \right\}$$

from the following result inspired by [25, Proposition 1].

Lemma 1. *Let Assumption 1 hold. Then, given the measurement $z(k) \in \mathbb{R}^{n_y}$ satisfying (4) with bounded noise $\gamma(k) \in \mathcal{Z}_\gamma = \langle c_\gamma, G_\gamma \rangle$, the corresponding state vector $x(k) \in \mathbb{R}^{n_x}$ is contained within the zonotope $\mathcal{Z}_{x|z(k)} = \langle c_{x|z(k)}, G_{x|z(k)} \rangle$, where*

$$\begin{aligned} c_{x|z(k)} &= H^\dagger \big(z(k) - c_\gamma \big) \\ G_{x|z(k)} &= \begin{bmatrix} H^\dagger G_\gamma & \kappa (I_{n_x} - H^\dagger H) \end{bmatrix}. \end{aligned} \tag{6}$$

Proof. By Assumption 1, we have $-\kappa 1_{n_x} \leq x(k) \leq \kappa 1_{n_x}$ element-wise. Thus, the zonotope $\mathcal{Z}_\kappa = \langle 0_{n_x}, \kappa I_{n_x} \rangle$ contains $x(k)$. Left-multiplying by $H^\dagger$, adding $x(k)$ on both sides, and rearranging (4) gives

$$x(k) = H^\dagger \left(z(k) - \gamma(k) \right) + (I_{n_x} - H^\dagger H) x(k) \tag{7}$$

which implies that $\mathcal{Z}_{x|z(k)} = H^\dagger \left(z(k) - \mathcal{Z}_\gamma \right) + (I_{n_x} - H^\dagger H) \mathcal{Z}_\kappa$. Therefore, performing the linear transformations and Minkowski sum operations gives $\mathcal{Z}_{x|z(k)}$ with the center $c_{x|z(k)}$ and the generator $G_{x|z(k)}$ as in (6). ☐

Using the zonotope $\mathcal{Z}_{x|z(k)}$ for each sample $x(k)$, we obtain a matrix zonotope that provides the mapping of Z^+ and Z^- to the state space. For $\gamma(k)$, consider the extension of the noise zonotopes $\mathcal{Z}_\gamma = \langle c_\gamma, [g_\gamma^{(1)} \dots g_\gamma^{(\xi_\gamma)}] \rangle$ to a matrix zonotope $\mathcal{M}_\gamma = \langle C_\gamma, \mathcal{G}_\gamma \rangle$, where $C_\gamma = [\; c_\gamma \; \dots \; c_\gamma \;] \in \mathbb{R}^{n_y \times T}$ and $\mathcal{G}_\gamma = \{ G_\gamma^{(1)}, \dots, G_\gamma^{(\xi_\gamma T)} \}$ with $G_\gamma^{(j+(i-1)T)} = [\, 0_{n_y \times (j-1)} \; g_\gamma^{(i)} \; 0_{n_y \times (T-j)} \,]$, for $i = 1, \dots, \xi_\gamma$ and $j = 1, \dots, T$ and similarly we define $\mathcal{M}_w = \langle C_w, \mathcal{G}_w \rangle$.

Proposition 1. Let Assumption 1 hold. Then, given the data Z^+ in (5), and noise matrix zonotope $\mathcal{M}_\gamma = \langle C_\gamma, \mathcal{G}_\gamma \rangle$, the unknown sequence of state $X^+ = [\, x(1) \dots x(T) \,]$ is contained within the zonotope $\mathcal{M}^+_{X|Z} = \langle C^+_{X|Z}, \mathcal{G}_{X|Z} \rangle$, where

$$\begin{aligned} C^+_{X|Z} &= H^\dagger (Z^+ - C_\gamma) \\ \mathcal{G}_{X|Z} &= \{ H^\dagger \mathcal{G}_\gamma, \kappa (I_{n_x} - H^\dagger H) 1_{n_x \times T} \}. \end{aligned} \tag{8}$$

Proof. From (7), we have $X^+ = H^\dagger (Z^+ - \Gamma^+) + (I - H^\dagger H) X^+$ where $\Gamma^+ = [\, \gamma(1) \dots \gamma(T) \,] \in \mathcal{M}_\gamma$. Due to Assumption 1, we have $X^+ \in \mathcal{M}_\kappa$ where $\mathcal{M}_\kappa$ is a matrix zonotope with center at 0 and one generator $\kappa 1_{n_x \times T}$. Therefore, we over-approximate X^+ by

$$\mathcal{M}^+_{X|Z} = H^\dagger (Z^+ - \mathcal{M}_\gamma) + (I - H^\dagger H) \mathcal{M}_\kappa.$$

Hence, by applying the linear transformations and performing the Minkowski sum of the matrix zonotopes on the right-hand side of the above equation, we obtain (8). ☐

Let $X^- = [\, x(0) \dots x(T-1) \,]$. We over-approximate X^- by $\mathcal{M}^-_{X|Z}$ by making use of Proposition 1 and Z^-. From (3), consider

$$\Omega(X^-, U^-) = [\, h(x(0), u(0)) \dots h(x(T-1), u(T-1)) \,] \tag{9}$$

to be a matrix in $\mathbb{R}^{m_a \times T}$ with state and input trajectories substituted in $h(x(k), u(k))$, for $k = 0, \dots, T-1$. By converting the matrix zonotope $\mathcal{M}^-_{X|Z}$ into interval matrix $\mathcal{I}^-_{X|Z} = \text{int}(\mathcal{M}^-_{X|Z})$ and substitute in (9), we obtain an interval matrix

$$\Omega(\mathcal{I}^-_{X|Z}, U^-) = \Big[h(\mathcal{I}^-_{X|Z}(:, 0), u(0)) \quad \dots \quad h(\mathcal{I}^-_{X|Z}(:, T-1), u(T-1)) \Big].$$

Proposition 2. The matrix zonotope

$$\mathcal{M}_{\Theta_p} = (\mathcal{M}^+_{X|Z} - \mathcal{M}_w) \Omega(\mathcal{I}^-_{X|Z}, U^-)^\dagger \tag{10}$$

contains all matrices Θ_p that are consistent with the data $\{Z, U^-\}$ and the process noise matrix zonotope $\mathcal{M}_w$.

Proof. From (2) and (3), we have $X^+ = \Theta_p \Omega(X^-, U^-) + W^-$ where $W^- = [\, w(0) \dots w(T-1) \,]$. The true Θ_p can be represented by a specific choice of $\beta^{(i)}$ in the matrix noise zonotope $\mathcal{M}_w$ which in turn results in the specific $W^- \in \mathcal{M}_w$. However, as we do not know the true W^-, we consider all matrices in the matrix zonotope $\mathcal{M}_w$ and compute the corresponding Θ_p. Moreover, we do not have access to X^+, X^-, so we over-approximate Θ_p by considering the matrix zonotope $\mathcal{M}^+_{X|Z}$ in (8) that bounds X^+ by Proposition 1. Finally, instead of $\Omega(X^-, U^-)$, we consider the interval matrix $\Omega(\mathcal{I}^-_{X|Z}, U^-)$ which gives (10). ☐

The above proposition proves that the matrix zonotope $\mathcal{M}_{\Theta_p}$ contains the true Θ_p, however, it might be conservative. To obtain a tighter bound on the true Θ_p, one might resort to prior knowledge about the system, such as bounds on the parameters and zero-pattern structure of Θ_p, and incorporate it in the estimated set by using a constrained matrix zonotope. Such knowledge can be attained by studying the physics of the system or the environment in which the system operates. It would be beneficial to make use of this side information to have less conservative estimated set bounds. We consider the proposed approach in [32] to incorporate prior information about the unknown coefficients like decoupled dynamics, partial knowledge, or prior bounds on entries of the unknown coefficients matrices. We consider

Algorithm 1 Set-based estimation of polynomial systems

Input: Data matrices $\{Z, U^-\}$ of the polynomial system (2), initial set $\mathcal{X}_0$, process noise zonotope $\mathcal{Z}_w$ and matrix zonotope $\mathcal{M}_w$, online input $u(k)$ and measurement $y(k)$, for $k = 0, \ldots, N-1$.
Output: Estimated sets $\mathcal{N}_{\Theta_p}$ and $\hat{\mathcal{R}}_k$

// Offline phase //
1: Obtain $\mathcal{M}^-_{X|Z}, \mathcal{M}^+_{X|Z}$ from Proposition 1
2: $\mathcal{I}^-_{X|Z} = \text{int}(\mathcal{M}^-_{X|Z})$
3: $\mathcal{M}_{\Theta_p} = (\mathcal{M}^+_{X|Z} - \mathcal{M}_w)\Omega(\mathcal{I}^-_{X|Z}, U^-)^\dagger$
4: $\overline{C}_{\Theta_p} = C_{\Theta_p}$
5: $\overline{G}^{(i)}_{\Theta_p} = G^{(i)}_{\Theta_p}$, for $i = 1, \ldots, \xi_{\Theta_p}$
6: $\overline{G}^{(i)}_{\Theta_p} = 0$, for $i = \xi_{\Theta_p} + 1, \ldots, \xi_{\Theta_p} + n_x m_a$
7: $\overline{\mathcal{G}}_{\Theta_p} = \{\overline{G}^{(1)}_{\Theta_p}, \ldots, \overline{G}^{(\xi_{\Theta_p} + n_x m_a)}_{\Theta_p}\}$
8: $A^{(i)}_{\Theta_p} = \bar{Q} G^{(i)}_{\Theta_p}$, for $i = 1, \ldots, \xi_{\Theta_p}$
9: $A^{(\xi T + k)}_{\Theta_p} = -E_i R E_j$, where $(i, j) \mapsto k = 1, \ldots, n_x m_a$, for $i = 1, \ldots, n_x$ and $j = 1, \ldots, m_a$
10: $\mathcal{A}_{\Theta_p} = \{A^{(1)}_{\Theta_p}, \ldots, A^{(\xi_{\Theta_p} + n_x m_a)}_{\Theta_p}\}$, $B_{\Theta_p} = \bar{Y} - \bar{Q} C_{\Theta_p}$
11: $\mathcal{N}_{\Theta_p} = \langle \overline{C}_{\Theta_p}, \overline{\mathcal{G}}_{\Theta_p}, \mathcal{A}_{\Theta_p}, B_{\Theta_p} \rangle$
// Online phase //
12: $\hat{\mathcal{R}}_0 = \mathcal{X}_0$
13: **for** $k = 0 : N - 1$ **do**
14: $\quad \hat{\mathcal{I}}_k = \text{int}(\hat{\mathcal{R}}_k)$
15: $\quad \tilde{\mathcal{R}}_{k+1} = \mathcal{N}_{\Theta_p} \Omega(\hat{\mathcal{I}}_k, u(k)) + \mathcal{Z}_w$
16: $\quad$ Obtain $\hat{\mathcal{R}}_{k+1}$ from Proposition 3
17: **end for**

any side information that can be formulated as

$$|\bar{Q}\Theta_p - \bar{Y}| \leq \bar{R} \tag{11}$$

where $\bar{Q} \in \mathbb{R}^{n_s \times n_x}$, $\bar{Y} \in \mathbb{R}^{n_s \times m_a}$, and $\bar{R} \in \mathbb{R}^{n_s \times m_a}$ are matrices defining the side information for the true $\Theta_p \in \mathbb{R}^{n_x \times m_a}$. Here, $|\cdot|$ and $\leq$ are element-wise operators.

After obtaining a matrix zonotope $\mathcal{M}_{\Theta_p}$ that bounds the set of unknown coefficients, we utilize the constrained matrix zonotopes to incorporate side information in the set-based estimation. Specifically, we compute the constrained matrix zonotope $\mathcal{N}_{\Theta_p}$ as in lines 4-11 of Algorithm 1. We use the same center of the $\mathcal{M}_{\Theta_p}$ in line 4 and append zeros to its list of generators in lines 5 and 6. Then, we compute the list of constrained matrices in lines 8 to 10. Our computations to tighten the set of the unknown coefficients are adaptations of the theory in [32] and can be easily proved.

B. Online Phase

We present the online estimation phase by considering the system (2) with measurements $y(k)$. This phase consists of a time update step computing $\tilde{\mathcal{R}}_k$ and a measurement update step computing $\hat{\mathcal{R}}_k$ as described in Algorithm 1.

1) Time update

We first initialize the measurement update set $\hat{\mathcal{R}}_0$ in line 12. Then, at each time step $k = 0, \ldots, N-1$, we convert the estimated set into an interval $\hat{\mathcal{I}}_k$ in line 14 by using (1). Given the current input $u(k)$ and the interval matrix $\hat{\mathcal{I}}_k$, we substitute in the list of monomials $\Omega(\hat{\mathcal{I}}_k, u(k))$. Then, in line 15, we propagate ahead the measurement update set $\hat{\mathcal{I}}_k$ using the constrained matrix zonotope $\mathcal{N}_{\Theta_p}$ obtained in the offline phase, interval of all monomials $\Omega(\hat{\mathcal{I}}_k, u(k))$, and the noise zonotope $\mathcal{Z}_w$. The computation in line 15 requires multiplying a constrained matrix zonotope $\mathcal{N}_{\Theta_p}$ by an interval $\Omega(\hat{\mathcal{I}}_k, u(k))$ and Minkowski sum with a zonotope $\mathcal{Z}_w$. This can be over-approximated by either converting $\Omega(\hat{\mathcal{I}}_k, u(k))$ to a matrix zonotope [32] and follow the traditional multiplication scheme in [31], [38] or by converting the $\Omega(\hat{\mathcal{I}}_k, u(k))$ to a zonotope and follow the multiplication proposed in [32, Proposition 2]. In general, the results of the computations can be represented as a constrained zonotope $\tilde{\mathcal{R}}_k = \langle \tilde{c}_k, \tilde{G}_k, \tilde{A}_k, \tilde{b}_k \rangle$.

2) Measurement update

We use the implicit intersection approach presented in [25], where the measurement update set $\hat{\mathcal{R}}_k$ is determined directly from the time update set $\tilde{\mathcal{R}}_k$ and the measurements $y(k)$ in line 16 as presented in the following proposition.

Proposition 3. ([25]) The intersection of $\tilde{\mathcal{R}}_k = \langle \tilde{c}_k, \tilde{G}_k, \tilde{A}_k, \tilde{b}_k \rangle$ and a region for $x(k)$ corresponding to $y(k)$ as in (2b) can be described by the constrained zonotope $\hat{\mathcal{R}}_k = \langle \hat{c}_k, \hat{G}_k, \hat{A}_k, \hat{b}_k \rangle$ where $\hat{c}_k = \tilde{c}_k$, $\hat{G}_k = \tilde{G}_k$, and

$$\hat{A}_k = \begin{bmatrix} \tilde{A}_k & 0 \\ H\tilde{G}_k & G_v \end{bmatrix}, \hat{b}_k = \begin{bmatrix} \tilde{b}_k \\ y(k) - Hc_k - c_v \end{bmatrix}.$$

The reachable set computed in Algorithm 1 over-approximates the exact reachable set, i.e., $\mathcal{R}_{k+1} \subseteq \hat{\mathcal{R}}_{k+1}$ due to the inclusion of the true Θ_p inside $\mathcal{M}_{\Theta_p}$ and accordingly inside $\mathcal{N}_{\Theta_p}$ under the assumption that the side information (11) holds for the true Θ_p.

IV. Application to SIR Epidemics

Consider a discrete-time SIR epidemic model with constant population

$$x_1(k+1) = x_1(k) - \beta x_1(k) x_2(k) + w_1(k)$$
$$x_2(k+1) = x_2(k) + \beta x_1(k) x_2(k) - \gamma x_2(k) + w_2(k)$$
$$x_3(k+1) = x_3(k) + \gamma x_2(k) + w_3(k)$$

where $k = 0, 1, 2, \ldots$ are days; $x_1(k), x_2(k), x_3(k) \in [0, 1]$ are respectively the proportions of susceptible, infected, and recovered populations; $\beta, \gamma \in [0, 1]$ are the infection and recovery parameters; and $w_1(k), w_2(k), w_3(k)$ are the bounded process noise inputs. Notice that the discretization step $\bar{h}$ is assumed to be one day, therefore, if $\beta \in [0, 1]$, then the condition $\bar{h}\beta \leq 1$ of [39] is satisfied. Considering

$$\Theta_p = \begin{bmatrix} 1 & 0 & 0 & -\beta \\ 0 & 1-\gamma & 0 & \beta \\ 0 & \gamma & 1 & 0 \end{bmatrix} \tag{12}$$

$$h(x(k)) = \begin{bmatrix} x_1(k) & x_2(k) & x_3(k) & x_1(k)x_2(k) \end{bmatrix}^\mathsf{T}$$

then, from (2) and (3), we can write the SIR model as

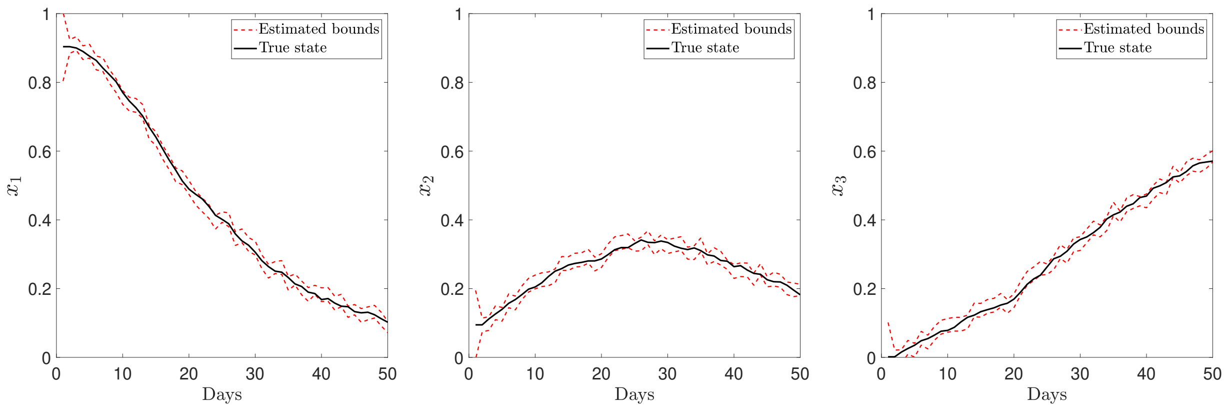
$$x(k+1) = f_p(x(k)) + w(k) = \Theta_p h(x(k)) + w(k)$$

where $x = [\, x_1 \;\, x_2 \;\, x_3 \,]^\mathsf{T}$ and $w = [\, w_1 \;\, w_2 \;\, w_3 \,]^\mathsf{T}$. The output

$$y(k) = \begin{bmatrix} x_2(k) \\ x_3(k) \\ x_1(k) + x_2(k) + x_3(k) \end{bmatrix} + \begin{bmatrix} v_1(k) \\ v_2(k) \\ v_3(k) \end{bmatrix}$$

TABLE I: System model bounds.

	Left bound	Right bound
Matrix zonotope $\mathcal{M}_{\Theta_p}$	$\begin{bmatrix} 0.23 & -0.17 & -0.23 & -0.70 \\ -1.29 & 0.63 & -0.35 & -1.20 \\ -1.61 & -0.41 & 0.58 & -1.30 \end{bmatrix}$	$\begin{bmatrix} 1.79 & 0.23 & 0.19 & 0.66 \\ 1.37 & 1.32 & 0.38 & 1.11 \\ 1.50 & 0.39 & 1.43 & 1.41 \end{bmatrix}$
Constrained matrix zonotope $\mathcal{N}_{\Theta_p}$	$\begin{bmatrix} 1 & 0 & 0 & -0.69 \\ 0 & 0.63 & 0 & 0 \\ 0 & -0.30 & 1 & 0 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1.30 & 0 & 1 \\ 0 & 0.39 & 1 & 0 \end{bmatrix}$



(a) Susceptible $x_1(k)$ (b) Infected $x_2(k)$ (c) Recovered $x_3(k)$

Fig. 1: Set-based state estimation of SIR epidemics.

where $v_1(k), v_2(k), v_3(k)$ are the bounded measurement noise variables. The last output $x_1(k) + x_2(k) + x_3(k) \approx 1$ is due to the constant population assumption. For the SIR epidemic model, we need to measure γI for observability and identifiability [40]. However, since γ is unknown, we need to measure at least two states to satisfy these properties.

We generate Z as in (5) using the true values of $\beta = 0.3$ and $\gamma = 0.1$, $x(0) = [\ 0.9 \quad 0.09 \quad 0.01\ ]^{\intercal}$, and $w(k) \in \langle 0, [\ 0.001 \quad 0.001 \quad 0.001\ ]^{\intercal} \rangle$ and $v(k) \in \langle 0, [\ 0.001 \quad 0.001 \quad 0.001\ ]^{\intercal} \rangle$. We use Z, without the knowledge of β and γ, to find the constrained matrix zonotope $\mathcal{N}_{\Theta_p}$ and ensuring that $\Theta_p \in \mathcal{N}_{\Theta_p}$. First, we find $\mathcal{M}_{\Theta_p}$ using (10), whose left and right bounds are given in Table I. Second, as the structure of Θ_p is given in (12), we use this prior knowledge and impose it on the matrix zonotope $\mathcal{M}_{\Theta_p}$ by using the inequality (11), where

$$\bar{Y} = \begin{bmatrix} 1 & 0 & 0 & -0.5 \\ 0 & 1-0.3 & 0 & 0.5 \\ 0 & 0.3 & 0 & 1 \end{bmatrix}, \quad \bar{R} = \begin{bmatrix} 0 & 0 & 0 & 0.5 \\ 0 & 0.5 & 0 & 0.5 \\ 0 & 0.5 & 0 & 0 \end{bmatrix}$$

are the initial guess on Θ_p and the maximum uncertainty on $\bar{Y}$, and $\bar{Q} = I_{n_x}$. The matrix $\bar{R}$ is chosen as such by keeping into account the prior knowledge that $\beta, \gamma \in [0, 1]$. From the lines 8–11 of Algorithm 1, we finally obtain $\mathcal{N}_{\Theta_p}$, whose left and right bounds are given in Table I. From $\mathcal{N}_{\Theta_p}$, we obtain $-0.69 \leq -\beta \leq 0$, which gives $\beta \in [0, 0.69]$, and $0.63 \leq 1 - \gamma \leq 1.30$, intersecting it with prior knowledge $\gamma \in [0, 1]$ gives $\gamma \in [0, 0.37]$. This validates Proposition 2 as Θ_p with true $\beta = 0.3$ and $\gamma = 0.1$ is contained in $\mathcal{N}_{\Theta_p}$.

For the online phase, we first suppose that the region $\mathcal{X}_0$ for the initial condition is known, where it is guessed that $x_1(0) \in [0.8, 1]$, $x_2(0) \in [0, 0.2]$, and $x_3(0) \in [0, 0.1]$. We first consider $\hat{\mathcal{R}}_0 = \mathcal{X}_0$, then, at each time step k, using the noisy online measurements $y(k)$, we update the set estimate $\hat{\mathcal{R}}_k$ using the lines 14–16 of Algorithm 1. These set-based estimation results are depicted in Figure 1, where it is ensured that the true state $x(k)$ remains bounded by the constrained zonotope $\hat{\mathcal{R}}_k$. This validates Proposition 3.

V. Conclusion

We presented a data-driven set-based estimator for polynomial systems. In the offline phase, a set of models is computed that is consistent with the experimental data and the noise bounds. In the online phase, the time update step involves propagating ahead of the estimated set using the set of consistent models. Then, the measurement update step computes the set-based estimate by intersecting the time update set with the set consistent with noisy measurements. We evaluate the proposed approach on SIR epidemics by estimating sets that bound the model parameters and the proportion of susceptible, infected, and recovered populations under bounded uncertainties and measurement noise.

Future investigations include the development of methods that further reduce the conservativeness of estimated sets without violating the formal guarantees. Moreover, observability conditions of polynomial systems need to be examined when the model is partially known.

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