Feedback Design for Devising Optimal Epidemic Control Policies

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Abstract: This paper proposes a feedback design that effectively copes with uncertainties for reliable epidemic monitoring and control. There are several optimization-based methods to estimate the parameters of an epidemic model by utilizing past reported data. However, due to the possibility of noise in the data, the estimated parameters may not be accurate, thereby exacerbating the model uncertainty. To address this issue, we provide an observer design that enables robust state estimation of epidemic processes, even in the presence of uncertain models and noisy measurements. Using the estimated model and state, we then devise optimal control policies by minimizing a predicted cost functional. To demonstrate the effectiveness of our approach, we implement it on a modified SIR epidemic model. The results show that our proposed method is efficient in mitigating the uncertainties that may arise.

1. INTRODUCTION

Since the onset of the SARS-CoV-2 outbreak, there has been a surge of interest in epidemic processes from many fields including the controls community. These works typically consider analysis, parameter identification, state estimation, forecasting, and/or control of a particular compartmental model that may or may not be networked, e.g., (Paré et al., 2020). Further, there is also a rich body of literature from the controls field prior to the COVID-19 Pandemic (Nowzari et al., 2016; Mei et al., 2017; Paré et al., 2018). In this work, we present a unified framework for parameter estimation, state estimation, and optimal control on a generic class of nonlinear models that includes most of the deterministic epidemic models.

The parameter and state estimation problems are questions of identifiability and observability, respectively. Conventionally, differential geometric techniques were employed for obtaining sufficient conditions that verify these notions for nonlinear systems (Grewal and Glover, 1976). On the other hand, to obtain necessary and sufficient conditions, Diop and Fliess (1991) introduced differential algebraic methods for identifiability and observability. The concepts were further developed and applied to biological models later (Audoly et al., 2001; Saccomani et al., 2003).

Once identifiability has been verified, the parameters must be estimated, for which several methods exist in the literature. Most common among them are the gradient-based and Newton-type methods like Levenberg–Marquardt and trust region reflective algorithms (Ljung, 1999). For some basic epidemic models, like SIR, explicit expressions for parameters are derived by Hadeler (2011) and Magal and Webb (2018). These ideas have also been explored for networked epidemics (Paré et al., 2020). However, if the state variables in those expressions cannot be directly measured, these techniques cannot be employed.

On the other hand, constrained optimal control is also a rich area of research with two techniques typically employed in practice. The first one is to use Pontryagin's minimum principle for computing the optimal control trajectory. However, in general, this principle is only a necessary condition of optimality. It is sufficient only in the case when the Hamiltonian functional is convex in the state variable. A practically superior method to solve these types of problems is to convert them to a constrained nonlinear optimization problem and use a numerical solver. Optimal control has been employed for epidemic mitigation before, e.g., (Köhler et al., 2021).

Unlike parameter estimation and optimal control, the literature on state estimation of epidemic processes is lacking. Designing robust observers to accurately estimate the current state of the epidemic model is the missing component in the feedback optimal control. However, due to being nonlinear, observer design for epidemic processes

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is quite challenging. Extended Kalman filtering techniques (Rajaei et al., 2021) are based on linearization and can only provide local guarantees. Therefore, one must know the initial state quite accurately to obtain a good state estimate when using these techniques. On the other hand, observer design techniques for general nonlinear systems with global guarantees turn out to be very conservative for epidemic models (Niazi and Johansson, 2022).

Our main contributions include the extension of the observer proposed by Niazi and Johansson (2022) and providing robust guarantees under uncertainties. Moreover, devising optimal epidemic policies using state estimates from the observer is another novelty. Our simulation results demonstrate that incorporating a robust observer in the feedback loop yields more reliable epidemic control.

The rest of the paper is organized as follows. Section 2 formulates the problem of interest. Section 3 outlines the proposed framework and Section 4 provides necessary background. Section 5 presents our proposed algorithm for robust state estimation and Section 6 demonstrates our method on a modified SIR epidemic model.

Notations. The Euclidean norm of $x \in \mathbb{R}^n$ is denoted as $\|x\| \doteq \sqrt{x^\top x}$. For a function $w \in L^\infty(\mathbb{R};\mathbb{R}^n)$, the essential supremum norm $\|w\|_\infty \doteq \operatorname{ess\,sup}_{t\in\mathbb{R}} \|w(t)\|$. By $w_{[t_0,t_1]}$, we denote the restriction of w to $[t_0, t_1]$ for some $t_1 > t_0$. The maximum singular value of $M \in \mathbb{R}^{n\times m}$ is denoted as $\sigma_{\max}(M)$. An identity matrix of size $n \times n$ is I_n. For $M \in \mathbb{R}^{n\times n}$, $\mathtt{sym}(M) \doteq M + M^\top$, and $M \geqslant 0$ ($M > 0$) means that M is positive semi-definite (resp., definite).

2. PROBLEM DEFINITION

We consider a class of deterministic epidemic models

$$\dot{x}(t) = Ax(t) + Gf(Hx(t), u(t)) \tag{1a}$$

$$y(t) = Cx(t) \tag{1b}$$

where $x(t) \in \mathcal{X} \subset \mathbb{R}^{n_x}$ is the state, $u(t) \in \mathcal{U} \subset \mathbb{R}^{n_u}$ is the control input, and $y(t) \in \mathbb{R}^{n_y}$ is the measured output. Each element $x_i(t)$ of the state vector corresponds to a different epidemic variable or compartment, and each element $y_i(t)$ of the output vector corresponds to a certain measurable epidemic variable. On the other hand, each element $u_i(t)$ of the input vector corresponds to a certain pharmaceutical or non-pharmaceutical interventions like improving medical facilities, testing and isolation, quarantining, vaccination, lockdown, social distancing, and travel restrictions, which can be enforced by a public authority.

We have $A \in \mathbb{R}^{n_x\times n_x}, G \in \mathbb{R}^{n_x\times n_g}, C \in \mathbb{R}^{n_y\times n_x}$ with

$$A \doteq A(\theta), \quad G \doteq G(\theta), \quad C \doteq C(\theta)$$

where θ is the vector of epidemic parameters. The matrix $H \in \{0,1\}^{n_H\times n_x}$ is known and specifies the state variables involved in the nonlinear function $f : \mathcal{X}\times\mathcal{U} \to \mathbb{R}^{n_f}$, where f is smooth and thus Lipschitz continuous on a compact domain $\mathcal{X}\times\mathcal{U}$. That is, for every $x, \hat{x} \in \mathcal{X}$ and $u \in \mathcal{U}$, there exists $\ell \geqslant 0$ such that

$$\|f(Hx, u) - f(H\hat{x}, u)\| \leqslant \ell\|Hx - H\hat{x}\| \tag{2}$$

where

$$\ell = \sup_{(x,u)\in\mathcal{X}\times\mathcal{U}} \sigma_{\max}\left(\frac{\partial f}{\partial x}(Hx, u)\right). \tag{3}$$

Note that f depends only on the state $x(t)$ and the input $u(t)$, and not on the parameters θ.

Remark 1. The class of nonlinear systems (1) captures a variety of epidemic models in the literature. For instance, all the basic SIS, SIR, SEIR models (Hethcote, 1989) and their variants (Giordano et al., 2021; Niazi et al., 2021) can be written in the form of (1a). The networked epidemic models (Paré et al., 2020) can also written as (1a). △

The reported data on a time interval $[t_0, t_1]$ is given by

$$\bar{u}(t) = u(t) + \delta_u(t) \tag{4a}$$

$$\bar{y}(t) = y(t) + \delta_y(t) \tag{4b}$$

where $t_0 \geqslant 0$ is the time of epidemic onset, $t_1 > t_0$ is the current time, and $\delta_u(t)$ and $\delta_y(t)$ represent the uncertainties in the input-output data. The uncertainties $\delta_u(t)$ and $\delta_y(t)$ are unknown and account for clerical errors and delays in recording and reporting the data.

Problem Statement. Given the input-output data $(\bar{u}, \bar{y})$ for the past time interval $[t_0, t_1]$, we first aim to estimate the parameters θ and the current state $x(t_1)$ of (1). Then, based on the estimated model, we devise an optimal control policy $u(t)$ for a future time interval $[t_1, t_2]$, $t_2 > t_1$, by minimizing a given cost functional

$$J(x_{[t_1,t_2]}, u_{[t_1,t_2]}) \doteq \int_{t_1}^{t_2} q(x, u, t)\mathrm{d}t \tag{5}$$

subject to a set of specified constraints

$$\begin{aligned} r_i(x_{[t_1,t_2]}, u_{[t_1,t_2]}) &= 0, \quad & i &= 1, 2, \ldots, k \\ s_j(x_{[t_1,t_2]}, u_{[t_1,t_2]}) &\leqslant 0, \quad & j &= 1, 2, \ldots, l \end{aligned} \tag{6}$$

where $x_{[t_1,t_2]} \doteq x(t, u; \hat{x}_{t_1}, \hat{\theta})$ is the predicted state trajectory obtained by integrating (1a) for $t \in [t_1, t_2]$ using the values of estimated parameters $\theta = \hat{\theta}$ and choosing the initial condition as the estimated state $x(t_1) = \hat{x}_{t_1}$.

3. OUTLINE OF THE PROPOSED METHOD

Given the past input-output data (4) and the cost functional (5) with constraints (6), the proposed feedback design for (1) has three main constituents.

(1) **Parameter estimation.** Given the past input-output data $(\bar{u}(t), \bar{y}(t))$, for $t \in [t_0, t_1]$, we estimate the model parameters θ by solving

$$\hat{\theta} = \arg\min_{\theta\in\Theta} \int_{t_0}^{t_1} \|\bar{y}(t) - y(t, \bar{u}; \theta)\|\mathrm{d}t \tag{7}$$

where $y(t, \bar{u}; \theta)$ is the output of (1) at time t given the parameters θ.

(2) **State estimation.** Given the past input-output data $(\bar{u}(t), \bar{y}(t))$, for $t \in [t_0, t_1]$, we design a state observer that estimates the current state $x(t_1)$ by using the estimated parameters $\hat{\theta}$ in (1).

(3) **Optimal control:** Given the estimated model (1) with parameters $\hat{\theta}$ and the state $\hat{x}(t_1)$, we solve

$$u^*(t) = \arg\min_{u\in\mathcal{U}} J(x, u) \tag{8a}$$

subject to (6) and

$$\begin{cases} \dot{x} = A(\hat{\theta})x + G(\hat{\theta})f(Hx, u) \\ t \in [t_1, t_2],\ x(t_1) = \hat{x}(t_1) \end{cases}$$

where the cost functional $J(x, u)$ is defined in (5).

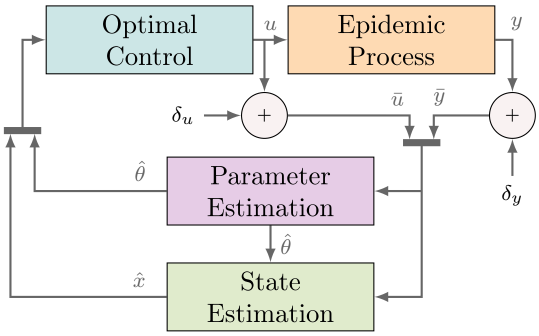

Fig. 1. Block scheme of optimal feedback control.

4. BACKGROUND MATERIAL

In this section, we introduce notions of identifiability and observability, and describe techniques to solve the parameter estimation and optimal control problems.

4.1 Verifying identifiability and observability

Identifiability is necessary for estimating model parameters from the input-output data $(\bar{u}, \bar{y})$. If the model has non-identifiable parameters, then it is not possible to estimate them uniquely (Saccomani et al., 2003).

Definition 2. System (1) is *locally identifiable* if, for almost any parameter vector $\theta \in \Theta$, there exists a neighborhood $\mathcal{N}(\theta) \subseteq \Theta$ such that, for every $\hat{\theta} \in \mathcal{N}(\theta)$,

$$y(t;\theta) = y(t;\hat{\theta}), \forall t \geqslant 0 \Leftrightarrow \theta = \hat{\theta}.$$

On the other hand, the notion of observability guarantees whether or not input-output data contains sufficient information about the system's state (Bernard et al., 2022).

Definition 3. System (1) is *locally observable* if, for any $\tau \geqslant 0$ and almost any $x(\tau) \in \mathcal{X}$, there exists a neighborhood $\mathcal{N}(x(\tau)) \in \mathcal{X}$ such that, for every $\hat{x}(\tau) \in \mathcal{N}(x(\tau))$,

$$y(t, u; x(\tau)) = y(t, u; \hat{x}(\tau)), \forall t \in [\tau, \infty) \Leftrightarrow x(\tau) = \hat{x}(\tau).$$

Identifiability and observability of (1) are required for the well-posedness of the parameter and state estimation problems, respectively. To verify these notions for (1), we employ the GenSSI algorithm proposed by Ligon et al. (2018), which tests the injectivity of the observation map obtained by taking Lie derivatives of the output.

4.2 Solving the parameter estimation problem

Once identifiability of the system has been verified, the parameters must be estimated. To carry out this process, optimization algorithms like Levenberg-Marquardt (Moré, 1978) and trust region (Byrd et al., 1987) can be used.

4.3 Solving the optimal control problem

In order to solve (8), we convert it to a constrained nonlinear optimization problem (Betts, 2010) and use interior point (Byrd et al., 1999) or trust region reflective (Coleman and Li, 1994) methods.

5. ALGORITHM FOR ROBUST STATE ESTIMATION

The main challenge in the proposed scheme (Figure 1) is the state estimation problem. Existing observer design techniques for the state estimation of epidemic processes are quite conservative and often turn out to be infeasible (Niazi and Johansson, 2022). Moreover, observers for epidemic models are often designed for specific compartmental models and cannot be adapted to other models. Here, we extend an observer for general epidemic models proposed by Niazi and Johansson (2022) to include model uncertainties and measurement noise.

After estimating the parameters from the data (4), (1) can be written as an uncertain nonlinear system

$$\dot{x}(t) = \hat{A}x(t) + \hat{G}f(Hx(t), \bar{u}(t)) + w(t) \quad (9a)$$

$$\bar{y}(t) = \hat{C}x(t) + v(t) \quad (9b)$$

where $w(t) \in \mathbb{R}^{n_x}$ is the model uncertainty, $v(t) \in \mathbb{R}^{n_y}$ is the measurement noise, and

$$\hat{A} \doteq A(\hat{\theta}), \quad \hat{G} \doteq G(\hat{\theta}), \quad \hat{C} \doteq C(\hat{\theta}).$$

Notice that the model uncertainty and measurement noise result from the uncertainties in the input-output data and the parameter estimation error $\theta - \hat{\theta}$.

Consider the observer proposed by Niazi and Johansson (2022):

$$\dot{z}(t) = Mz(t) + (ML + J)\bar{y}(t) + N\hat{G}f(q(t), \bar{u}(t)) \quad (10a)$$

$$\hat{x}(t) = z(t) + L\bar{y}(t) \quad (10b)$$

$$\hat{y}(t) = \hat{C}\hat{x}(t) \quad (10c)$$

where $q(t) \doteq H\hat{x}(t) + K(\bar{y}(t) - \hat{y}(t))$, $J, L \in \mathbb{R}^{n_x \times n_y}$ and $K \in \mathbb{R}^{n_H \times n_y}$ are matrices to be designed, and

$$M = \hat{A} - L\hat{C}\hat{A} - J\hat{C}, \quad N = I_{n_x} - L\hat{C}.$$

Here, $z(t) \in \mathbb{R}^{n_x}$ is the observer's state, and $\hat{x}(t) \in \mathbb{R}^{n_x}$ and $\hat{y}(t) \in \mathbb{R}^{n_y}$ are the state and output estimate of (9).

Consider the semidefinite programming (SDP) problem:

$$\text{minimize } \mu \text{ subject to} \quad (11a)$$

$$\begin{bmatrix} \mathtt{sym}(P\hat{A} - R\hat{C}\hat{A} - S\hat{C}) + Q & (P - R\hat{C})\hat{G} \\ G^\top (P - R\hat{C})^\top & -I_{n_f} \end{bmatrix} < 0 \quad (11b)$$

$$\begin{bmatrix} -Q & (H - K\hat{C})^\top \\ H - K\hat{C} & -\frac{1}{\ell^2} I_{n_H} \end{bmatrix} \leqslant 0 \quad (11c)$$

$$\begin{bmatrix} -\mu I_{n_x} & R \\ R^\top & -I_{n_y} \end{bmatrix} \leqslant 0 \quad (11d)$$

$$P = P^\top > 0 \text{ and } Q = Q^\top > 0 \quad (11e)$$

where ℓ is the Lipschitz constant obtained from (3), and

$$J = P^{-1}S, \quad L = P^{-1}R.$$

Theorem 4. If the SDP problem (11) is feasible, then there exist a $\mathcal{KL}$ function β and $\mathcal{K}_\infty$ functions α_1, α_2 such that the estimation error satisfies

$$\|x(t_1) - \hat{x}(t_1)\| \leqslant \beta(\|x(t_0) - \hat{x}(t_0)\|, t_1) + \alpha_1(\|w_{[t_0,t_1]}\|_\infty) + \alpha_2(\|v_{[t_0,t_1]}\|_\infty).$$

See (**?**) for the proof.

The above theorem implies that the state estimation error is stable with respect to the data and parameter uncertainties. Particularly, the noise attenuation is directly related to the parameter μ in (11). Moreover, in the absence of uncertainties, the estimate $\hat{x}(t_1)$ asymptotically converges to the true $x(t_1)$ given that t_1 is sufficiently large. As a result of the $\mathcal{KL}$ function β, the transient error resulting from the poor choice of $\hat{x}(t_0)$ also converges to zero asymptotically.

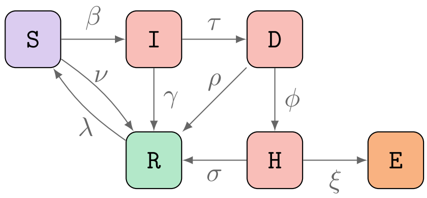

Fig. 2. Block diagram of SIDHER epidemic model.

Table 1. Description of unitless control inputs.

Control	Description
u_1	Stringency of NPIs
u_2	Proportion of medical resources dedicated
u_3	Testing capacity per population
u_4	Vaccination capacity per population

Table 2. Description of parameters.

Parameter	Description	Unit
β	Infection rate	1/day
γ	Recovery rate of undetected cases	1/day
ρ	Recovery rate of detected cases	1/day
σ	Recovery rate of hospitalized cases	1/day
ξ	Mortality rate of hospitalized cases	1/day
λ	Rate at which people lose immunity	1/day
ϕ	Hospitalization rate	1/day
τ	Testing rate of undetected cases	1/day
ν	Vaccination rate of susceptible cases	1/day

6. APPLICATION TO SIDHER EPIDEMIC MODEL

In this paper, we demonstrate the proposed method on an SIDHER epidemic model. After providing the model design, we implement the proposed method step-by-step and provide the simulation results.

6.1 Model design

We consider an SIDHER epidemic model (Susceptible, Infected, Detected, Hospitalized, Extinct, and Recovered), which is illustrated in Figure 2 and given by

$$\dot{\mathtt{S}}(t) = \lambda\mathtt{R}(t) - \beta\mathtt{S}(t)\mathtt{I}(t)(1-u_1(t)) - \nu\mathtt{S}(t)u_4(t) \quad (12a)$$

$$\dot{\mathtt{I}}(t) = -\gamma\mathtt{I}(t) + \beta\mathtt{S}(t)\mathtt{I}(t)(1-u_1(t)) - \tau\mathtt{I}(t)u_3(t) \quad (12b)$$

$$\dot{\mathtt{D}}(t) = -(\rho+\phi)\mathtt{D}(t) + \tau\mathtt{I}(t)u_3(t) \quad (12c)$$

$$\dot{\mathtt{H}}(t) = -\xi\mathtt{H}(t)(1-u_2(t)) + \phi\mathtt{D}(t) - \sigma\mathtt{H}(t)u_2(t) \quad (12d)$$

$$\dot{\mathtt{E}}(t) = \xi\mathtt{H}(t)(1-u_2(t)) \quad (12e)$$

$$\dot{\mathtt{R}}(t) = -\lambda\mathtt{R}(t) + \gamma\mathtt{I}(t) + \rho\mathtt{D}(t) + \nu\mathtt{S}(t)u_4(t) + \sigma\mathtt{H}(t)u_2(t) \quad (12f)$$

where all the state variables $\mathtt{S}(t), \mathtt{I}(t), \mathtt{D}(t), \mathtt{H}(t), \mathtt{E}(t), \mathtt{R}(t) \in [0,1]$, control input $u(t) = [\, u_1(t)\ u_2(t)\ u_3(t)\ u_4(t) \,]^\top$ explained in Table 1, and the parameters are in Table 2. Note that, for every $t \geqslant 0$,

$$\mathtt{S}(t) + \mathtt{I}(t) + \mathtt{D}(t) + \mathtt{H}(t) + \mathtt{E}(t) + \mathtt{R}(t) = 1. \quad (13)$$

Measured outputs. The model outputs are the following:

- $y_1 = \nu\mathtt{S}$: Since we measure the proportion of population vaccinated per day $\nu\mathtt{S}u_4$ and the vaccination capacity u_4 is known, we obtain the output y_1 by dividing both.
- $y_2 = \tau\mathtt{I}$: Since we measure the proportion of population tested per day $\tau\mathtt{I}u_3$ and the testing capacity u_3 is known, we obtain the output y_2 by dividing both.
- $y_3 = \mathtt{D}$: Active number of detected infected cases.
- $y_4 = \rho\mathtt{D}$: Daily number of cases recovering after detection.
- $y_5 = \phi\mathtt{D}$: Daily number of cases hospitalized after being detected.
- $y_6 = \mathtt{H}$: Active number of hospitalized cases.
- $y_7 = \sigma\mathtt{H}$: Daily number of cases recovering after hospitalization.
- $y_8 = \xi\mathtt{H}$: Daily number of deaths.
- $y_9 = \mathtt{E}$: Total number of deaths.
- $y_{10} = \mathtt{S} + \mathtt{I} + \mathtt{R}$: Since (13) holds and $\mathtt{D}, \mathtt{H}, \mathtt{E}$ are measured, we can obtain the output y_{10} by subtracting $\mathtt{D} + \mathtt{H} + \mathtt{E}$ from 1.

Model in vector form. We can write the model (12) in the form (1) with $f(Hx, u) = [\, \mathtt{SI}\ \mathtt{SI}u_1\ \mathtt{H}u_2\ \mathtt{I}u_3\ \mathtt{S}u_4 \,]^\top$ and the corresponding (A, C) pair is an observable (see (**?**) for the matrices and more details on the simulation setup and assumptions).

Identifiability and observability. Identifiability and observability of (12) is verified by the GenSSI software (Ligon et al., 2018) in MATLAB. See Fig. 3 for the zero-pattern structure of the Jacobian of the observability and identifiability map. The black boxes in the figure represent nonzero terms of the Jacobian, whereas white area represents zero terms. From the figure, it can be seen that the Jacobian has full generic rank, thus implying structural identifiability and observability. Therefore, it can be inferred that for almost all values of the initial states and parameters, (12) is at least locally identifiable and observable.

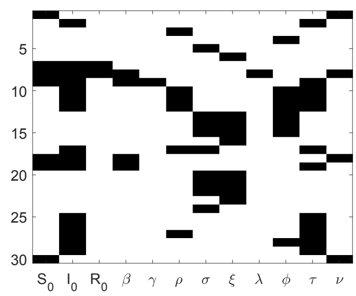

Fig. 3. Observability and identifiability tableau.

6.2 Simulation results

Data generation. We generate noisy synthetic data by simulating the model (12) for $t \in [t_0, t_1]$, where $t_0 = 0$ and $t_1 = 30$ days, with "true" parameters provided in Table 3 to illustrate the proposed method. The input-output data $(\bar{u}, \bar{y})$ is corrupted with white Gaussian noise that is sampled from $\mathcal{N}(0, 10^{-6})$. The nominal input is chosen as

$$u(t) = \begin{bmatrix} 0.01\lceil\sin(t/2)\rfloor + 0.015 \\ 0.01\lceil\cos(t/2)\rfloor + 0.015 \\ 0.01\lceil\sin(t/3)\rfloor + 0.015 \\ 0.01\lceil\cos(t/3)\rfloor + 0.015 \end{bmatrix}.$$

The true initial state is chosen to be

$$x_{t_0} = [\,0.999\;\; 10^{-3}\;\; 10^{-6}\;\; 0\;\; 0\;\; 0\,]^\top$$

which is only used for data generation and is not known by the parameter and state estimation algorithms.

Parameter estimation. The parameters ρ, ϕ, σ, ξ can be estimated using the least square solution by

$$\hat{\rho} = \int_{t_0}^{t_1} \frac{\bar{y}_3(t)^\top \bar{y}_4(t)}{\bar{y}_3(t)^\top \bar{y}_3(t)} \mathrm{d}t, \qquad \hat{\phi} = \int_{t_0}^{t_1} \frac{\bar{y}_3(t)^\top \bar{y}_5(t)}{\bar{y}_3(t)^\top \bar{y}_3(t)} \mathrm{d}t$$

$$\hat{\sigma} = \int_{t_0}^{t_1} \frac{\bar{y}_6(t)^\top \bar{y}_7(t)}{\bar{y}_6(t)^\top \bar{y}_6(t)} \mathrm{d}t, \qquad \hat{\xi} = \int_{t_0}^{t_1} \frac{\bar{y}_6(t)^\top \bar{y}_8(t)}{\bar{y}_6(t)^\top \bar{y}_6(t)} \mathrm{d}t.$$

The data $\bar{y}$ is discrete and can be interpolated for computing the above integrals. By fixing these estimated parameters, the remaining parameters are estimated using the `nlgreyest` function from the System Identification Toolbox in MATLAB employing the Trust-Region-Reflective Algorithm. See Table 3 for the estimated parameters. Notice that, for parameter estimation, we do not know the true initial state x_{t_0}. Instead, we guess the initial state appropriately, where $\mathtt{S}_{t_0}$ is chosen uniformly at random from $[0.95, 1]$; $\mathtt{I}_{t_0}$ from $[0, 0.05]$; $\mathtt{D}_{t_0}, \mathtt{H}_{t_0}, \mathtt{E}_{t_0}$ are obtained from $y_3(t_0), y_6(t_0), y_9(t_0)$; and $\mathtt{R}_{t_0}$ is obtained from (13).

Table 3. Model parameters.

Parameter	**True value**	**Estimated value**
β	0.3500	0.3530
γ	0.1000	0.0981
ρ	0.0500	0.0501
σ	0.0400	0.0399
ξ	0.0200	0.0202
λ	0.0167	0.0383
ϕ	0.1429	0.1428
τ	0.3000	0.2757
ν	0.0100	0.0100

Note that since the system (12) is locally identifiable, and the parameter estimation problem (7) is non-convex, the solution obtained in Table 3 might be local.

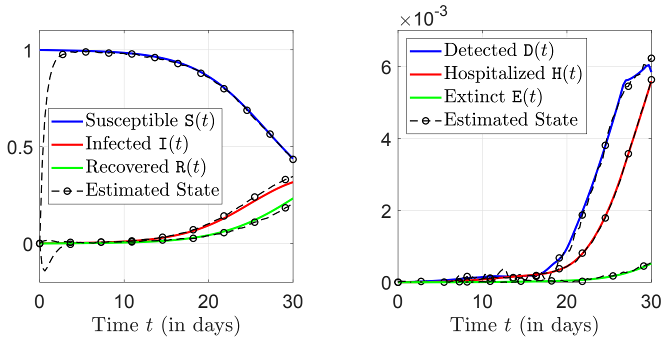

Fig. 4. State estimation by the proposed observer.

State estimation. After obtaining the estimated parameters, we employ the observer (10) for estimating the state of (12). The observer is designed by solving (11), which yields the design matrices J, K, L. The result of the state estimation method is demonstrated in Fig. 4. It can be seen that the observer converges very closely to the true state within three days of the epidemic outbreak.

Optimal control. In the optimal control problem, we choose the cost functional (5) as

$$J(x, u) = \int_{t_1}^{t_2} \left(x(t)^\top \Gamma x(t) + u(t)^\top \Lambda u(t) \right) \mathrm{d}t$$

where $\Gamma \geqslant 0$ and $\Lambda > 0$ are chosen to be

$$\Gamma = \mathtt{diag}(0.01, 1, 0, 2, 10, 0)$$
$$\Lambda = \mathtt{diag}(0.01, 0.01, 0.01, 0.01).$$

In short, we would like to minimize the susceptible, infected, hospitalized, and extinct cases in the time interval $[t_1, t_2]$. The control inputs are bounded as $0 \leqslant u_1(t) \leqslant 1$, $0 \leqslant u_2(t) \leqslant 0.9$, $0.1 \leqslant u_3(t) \leqslant 0.7$, and $0 \leqslant u_4(t) \leqslant 0.7$.

The equality constraint in (6) is (12) with estimated parameters (Table 3) and initialized at estimated state $\hat{x}(t_1)$. The inequality constraints in (6) are

$$\mathtt{I}(t) - \bar{\mathtt{I}} \leqslant 0,\; \mathtt{H}(t) - \bar{\mathtt{H}} \leqslant 0,\; \mathtt{E}(t) - \bar{\mathtt{E}} \leqslant 0$$

where we choose $\bar{I} = 0.5$, $\bar{\mathtt{H}} = 0.05$, and $\bar{\mathtt{E}} = 0.005$.

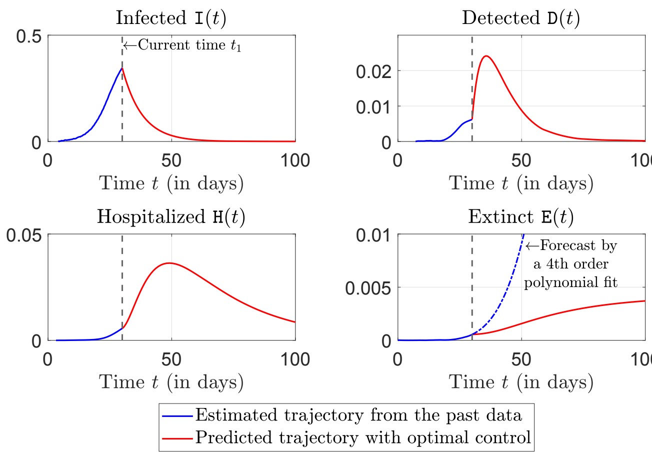

Fig. 5. State trajectory with optimal control input.

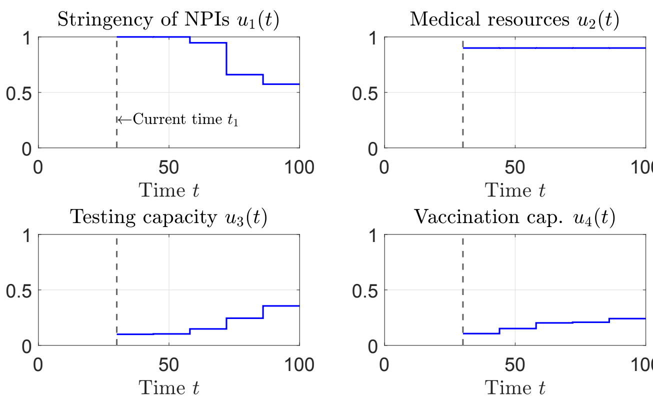

Fig. 6. Sequence of optimal control input.

We use the `fmincon` solver in MATLAB for obtaining the optimal solution of (8). We consider a piecewise constant control input $u(t)$, where $u(t)$ remains constant for a period of 14 days. The optimal control sequence is obtained for five such periods, i.e., the solution recommends how the policy should be varied in the future after every 14 days. The obtained optimal control input is shown in Fig. 6 and the resulting predicted state trajectory in Fig. 5. Notice from Fig. 5 that the constraints on I, H, and E are satisfied. From the forecast of E, using the polynomial fit of the estimated trajectory, it can be seen that the number of deaths are significantly reduced under the optimal control algorithm. This prediction is reliable only when, in addition to the parameters, the current state $x(t_1)$ is accurately estimated. The results may vary significantly if the unmeasured states are not chosen accurately. From Fig. 6, we can see that the optimal controller recommends to enact a full lockdown for the first four weeks to suppress the epidemic growth ($\dot{\mathtt{I}} < 0$). The lockdown is then lifted gradually by increasing the testing and vaccination capacities. On the other hand, to avoid the number of deaths and to satisfy the hard constraint $\mathtt{E}(t) \leqslant \bar{\mathtt{E}}$, the optimal controller recommends to employ full medical resources throughout the finite future horizon.

7. CONCLUDING REMARKS

We presented a unified framework for feedback optimal control of epidemic processes via a general class of nonlinear compartmental models. For the parameter estimation and constrained optimal control, we employed existing methods and techniques from system identification and optimal control theory. For the state estimation, however, the proposed robust observer design criteria is a novel contribution of this paper. By considering a realistic epidemic model, we demonstrated how optimal control policies can be devised by employing the proposed feedback design.

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