Clustering-Based Model Reduction of Networked Passive Systems

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***Abstract*—The model reduction problem for networks of interconnected dynamical systems is studied in this paper. In particular, networks of identical passive subsystems, which are coupled according to a tree topology, are considered. For such networked systems, reduction is performed by clustering subsystems that show similar behavior and subsequently aggregating their states, leading to a reduced-order networked system that allows for an insightful physical interpretation. The clusters are chosen on the basis of the analysis of controllability and observability properties of associated edge systems, representing the importance of the couplings and providing a measure of the similarity of the behavior of neighboring subsystems. This reduction procedure is shown to preserve synchronization properties (i.e., the convergence of the subsystem trajectories to each other) and allows for the *a priori* computation of a bound on the reduction error with respect to external inputs and outputs. The method is illustrated by means of an example of a thermal model of a building.**

***Index Terms*—Clustering, model reduction, multiagent systems, networks.**

I. Introduction

LARGE-SCALE networks of interconnected dynamical systems appear abundantly in both technology and nature, with examples ranging from electrical power grids to biological or chemical networks. Other examples are sensor networks, social networks and multiagent systems (see, e.g., [22], [35] for an overview). However, the large scale of such networked systems complicates its analysis or control, motivating the need for reduction techniques capable of finding *approximate* networked systems of lower complexity.

Existing model reduction techniques such as balanced truncation [25], optimal Hankel norm approximation [15] or moment matching [3] provide tools for the approximation of linear dynamical systems, hereby generating a reduced-order system (i.e., a system with lower state-space dimension) whose input-output behavior approximates that of the original system (see [1] for an overview). Even though such methods can have strong properties such as the preservation of stability properties or the availability of a computable error bound, they are not necessarily suited for the reduction of large-scale networked systems. Namely, such existing reduction techniques do generally not preserve the interconnection structure of a given networked system. As a result, the physical meaning of the states is not preserved and the resulting reduced-order system is difficult to interpret, which could for example complicate the synthesis of (distributed) controllers. The objective of this paper is therefore the development of a dedicated reduction procedure for networked systems, hereby approximating the input-output behavior of such a system while allowing for a physical interpretation of the reduced-order model.

The problem of the reduction of networked systems has received interest in fields ranging from economics [34] to chemistry [29] and biology [28]. However, none of these methods consider external inputs to the networked system, whereas only the latter considers external outputs. A systems theoretic perspective is taken in [18], where a reduced-order networked system is obtained by clustering scalar first-order subsystems that show similar behavior and subsequently aggregating the states of these subsystems. Moreover, the networked system has external inputs and a bound on the error in the state trajectories is given. Similarly, [24] gives a graph-partitioning approach towards clustering with the same properties. Early work on a clustering approach on the basis of a separation of time scales is presented in [10], whereas an approach for the identification of strongly coupled clusters in a network of dynamical systems is given in [33]. The idea of clustering is also exploited in the coherency-based reduction of power systems, see, e.g., [9], [14]. A different approach is taken in [23], where a network of identical higher-order linear subsystems is considered. Here, rather than performing reduction by clustering subsystems, model reduction is performed on the subsystems. Thus, the interconnection topology remains unaffected and such an approach can be interpreted as a structure-preserving model reduction procedure (see [31]).

In the current paper, the former approach is taken and a clustering-based model reduction procedure for networked systems is developed. Contrary to [18], subsystems with higher-order linear dynamics are considered and the networked system is assumed to have both external inputs and outputs. Specifically, the identical linear subsystems are assumed to be passive, whereas the (directed and weighted) interconnection topology is assumed to have a tree topology. It is remarked that passive systems form a physically relevant class of systems and that the passivity property provides a suitable tool for the analysis of networked systems (see, e.g., [2], [26]).

For such networked systems, a reduced-order system is obtained by clustering neighboring subsystems (which form the vertices of the network graph) and aggregating their states

Manuscript received February 23, 2015; revised August 17, 2015; accepted November 23, 2015. Date of publication December 9, 2015; date of current version September 23, 2016. This work was supported by the Knut and Alice Wallenberg Foundation, the Swedish Research Council VR under grant 2013-5523, and the Swedish Foundation for Strategic Research through the ICT-Psi project. Recommended by Associate Editor D. Bauso.

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Digital Object Identifier 10.1109/TAC.2015.2505418

to obtain a single subsystem that represents the original cluster. This thus essentially generates an updated interconnection topology which has a clear physical interpretation. In order to identify the neighboring subsystems most suitable for clustering, the controllability and observability properties of associated edge systems are analyzed. These properties, that will be referred to as *edge controllability* and *edge observability*, characterize how difficult it is to steer two neighboring subsystems apart or how hard it is to distinguish them, respectively. This analysis crucially relies on two aspects.

First, a novel factorization of the graph Laplacian (describing the interconnection topology) is introduced, which allows for the definition of a suitable *edge Laplacian* for directed and weighted graphs and facilitates the definitions of the edge systems mentioned above. Here, it is noted that this factorization extends a result for undirected graphs in [39]. For tree structures, it will be shown that this edge Laplacian has desirable properties in the scope of clustering-based model reduction for networked systems.

Second, motivated by a result in [6], the passivity property is exploited. Namely, this property allows for a decomposition of the (generalized) controllability and observability Gramians of the networked system into two parts, which are associated with the interconnection topology and the dynamics of the subsystems, respectively. By exploiting this decomposition, the influence of the interconnection topology on the controllability (observability) properties can be studied independently, providing a suitable basis for the selection of the neighboring subsystems to be clustered.

The main contribution of this paper is thus the development of a dedicated model reduction procedure for networked systems, for which the following properties are shown. First, it is shown that the characterization of edge controllability and edge observability is guaranteed to exist. Second, the synchronization property (i.e., the property that states of the subsystems converge to each other) is preserved during reduction. Third, a computable *a priori* bound on the reduction error (in terms of input-output behavior) is derived, providing a direct measure of the quality of the approximation and yielding a theoretical basis for the edge controllability and edge observability approach taken in the paper.

Finally, it is remarked that preliminary results on this topic can be found in [5].

The remainder of this paper is organized as follows. The problem setting is stated in Section II and preliminaries regarding passivity and graph theory are given in Section III. Then, the relation between edge systems and synchronization properties is discussed in Section IV, before presenting the clustering-based model reduction procedure in Section V. Finally, the procedure is illustrated by means of an example in Section VI and conclusions are stated in Section VII.

Notation: The field of real numbers is denoted by $\mathbb{R}$. For a vector $x \in \mathbb{R}^n$, the Euclidian norm is given as $|x| = \sqrt{x^T x}$. Next, given a matrix $X \in \mathbb{R}^{n\times m}$, its entry in row i and column j is denoted as $(X)_{ij}$ and $X \succ 0$ ($X \succcurlyeq 0$) denotes a positive (semi)definite matrix. The identity matrix of size n is denoted as I_n, whereas $\mathbf{1}_n$ denotes the vector of all ones of length n. The subscript n is omitted when no confusion arises. Moreover, e_i denotes the i-th column of I_n. A signal $x : \mathcal{T} \to \mathbb{R}^n$ is said to be in $\mathcal{L}_2^n(\mathcal{T})$ if $\int_{\mathcal{T}} |x(t)|^2 \mathrm{d}t < \infty$, where the corresponding norm for $\mathcal{T} = [0,\infty)$ is denoted as $\|x\|_2$. Finally, $X \otimes Y$ denotes the Kronecker product of matrices X and Y, whose definition and properties can be found in [7].

II. Problem Setting

A network of identical subsystems $\mathbf{\Sigma}_i$ is considered, of which the linear time-invariant dynamics is given as

$$\mathbf{\Sigma}_i : \begin{cases} \dot{x}_i = Ax_i + Bv_i \\ z_i = Cx_i \end{cases} \tag{1}$$

with $x_i \in \mathbb{R}^n$, $v_i, z_i \in \mathbb{R}^m$ and $i \in \{1, 2, \ldots, \bar{n}\}$. It is noted that the number of inputs v_i to (1) equals the number of outputs z_i, which allows for the interconnection of the subsystems $\mathbf{\Sigma}_i$ as

$$v_i = \sum_{j=1, j\neq i}^{\bar{n}} w_{ij}(z_j - z_i) + \sum_{j=1}^{\bar{m}} g_{ij} u_j \tag{2}$$

where $i \in \{1, 2, \ldots, \bar{n}\}$ and with $u_j \in \mathbb{R}^m$, $j \in \{1, 2, \ldots, \bar{m}\}$ the external inputs to the networked system. In (2), the weights $w_{ij} \in \mathbb{R}$ satisfying $w_{ij} \geq 0$ represent the strength of the diffusive coupling between the subsystems, whereas $g_{ij} \in \mathbb{R}$ describe the distribution and strength of the external inputs amongst the subsystems. After defining the weighted Laplacian matrix L as

$$(L)_{ij} = \begin{cases} -w_{ij}, & i \neq j \\ \sum_{j=1, j\neq i}^{\bar{n}} w_{ij}, & i = j \end{cases} \tag{3}$$

and collecting the parameters g_{ij} as in a matrix G as $(G)_{ij} = g_{ij}$, the interconnection (2) can be written as

$$v = -(L \otimes I_m)z + (G \otimes I_m)u \tag{4}$$

where $v^T = [v_1^T \; v_2^T \; \ldots \; v_{\bar{n}}^T]$, $z^T = [z_1^T \; z_2^T \; \ldots \; z_{\bar{n}}^T]$ and $u^T = [u_1^T \; u_2^T \; \ldots \; u_{\bar{m}}^T]$. The interconnection (4) as characterized through the matrix L can be associated to a graph, as will be discussed in more detail in Section III.

Similar to the external inputs introduced in (2), external outputs are given by

$$y_i = \sum_{j=1}^{\bar{n}} h_{ij} z_j \tag{5}$$

with $y_i \in \mathbb{R}^m$, $i \in \{1, 2, \ldots, \bar{p}\}$. Collecting the parameters $h_{ij} \in \mathbb{R}$ as $(H)_{ij} = h_{ij}$, this can be written as $y = (H \otimes I_m)z$, with $y^T = [y_1^T \; y_2^T \; \ldots \; y_{\bar{p}}^T]$. Then, combining the subsystem dynamics (1) with the interconnection (4) and the outputs leads to the dynamics of the networked system as

$$\mathbf{\Sigma} : \begin{cases} \dot{x} = (I \otimes A - L \otimes BC)x + (G \otimes B)u \\ y = (H \otimes C)x \end{cases} \tag{6}$$

where $x^T = [x_1^T \; x_2^T \; \ldots \; x_{\bar{n}}^T] \in \mathbb{R}^{\bar{n}n}$.

Given a networked system $\mathbf{\Sigma}$ as in (6), the objective of this paper is to find a reduced-order networked system that approximates the input-output behavior of $\mathbf{\Sigma}$ and allows for a suitable physical interpretation. To satisfy the latter aspect, a clustering-based reduction procedure is pursued, in which a reduced-order networked system is obtained by aggregating the states of neighboring subsystems. This essentially creates a new interconnection topology, which can be directly interpreted in terms of the original network structure. The reduced-order networked system obtained in this way should preserve synchronization properties (i.e., the property that the subsystem states converge to each other) and provide a bound on the reduction error.

III. Preliminaries

The reduction technique for networked systems developed in this paper will exploit results from graph theory, of which relevant definitions and results are given in this section. Moreover, the subsystems $\mathbf{\Sigma}_i$ will be assumed to be passive (see, e.g., [8], [37]).

Definition 1: A system $\mathbf{\Sigma}_i$ as in (1) is said to be *passive* if there exists a differentiable storage function $V : \mathbb{R}^n \to \mathbb{R}$ satisfying $V \geq 0$ and $V(0) = 0$ such that

$$\dot{V}(x_i) := \frac{\partial V}{\partial x_i}(x_i)\dot{x}_i \leq v_i^T z_i \tag{7}$$

holds along all trajectories of (1). If equality holds in (7), the system $\mathbf{\Sigma}_i$ is said to be *lossless*.

Passive systems can be written in port-Hamiltonian form [32], [38] as

$$\mathbf{\Sigma}_i : \begin{cases} \dot{x}_i = (J-R)Qx_i + Bv_i \\ z_i = B^T Q x_i \end{cases} \tag{8}$$

such that the matrices A and C in (1) are given as $A = (J-R)Q$ and $C = B^T Q$. In (8), $Q = Q^T$ characterizes the storage function V in (7) as $V(x_i) = (1/2)x_i^T Q x_i$, describing the energy stored in the system and satisfying $Q \succ 0$ if (8) is a minimal realization (see [38]). Moreover, J is a skew-symmetric matrix (i.e., $J = -J^T$) and $R = R^T \succcurlyeq 0$ characterizes the internal dissipation, such that $R = 0$ for lossless systems.

The interconnection (2) of the subsystems as characterized by the Laplacian matrix L in (3) can be associated to a directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ (see, e.g, [16], [22] for details on graph theory). Here, $\mathcal{V} = \{1, 2, \ldots, \bar{n}\}$ represents the set of vertices characterizing the subsystems and $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ gives the set of directed edges (or arcs) satisfying $(i,j) \in \mathcal{E}$ if and only if $w_{ji} > 0$.

Besides this directed graph $\mathcal{G}$, an undirected version of the same graph is introduced as follows.

Definition 2: Let $\mathcal{G}$ be a directed graph with vertex set $\mathcal{V}$ and (directed) edge set $\mathcal{E}$. Then, the undirected graph $\mathcal{G}_u = (\mathcal{V}, \mathcal{E}_u)$ with $(i,j) \in \mathcal{E}_u$ if and only if $w_{ij} + w_{ji} > 0$ is said to be the underlying undirected graph.

The underlying undirected graph $\mathcal{G}_u$ thus has an edge between vertices i and j if at least one of the weights w_{ij} and w_{ji} is strictly positive, i.e., if there exists at least one directed edge between i and j. Next, let $E \in \mathbb{R}^{\bar{n} \times \bar{n}_e}$ be the incidence matrix of an arbitrary orientation of $\mathcal{G}_u$ (with $\bar{n}_e$ the number of edges). Thus, if the l-th edge connects vertices i and j, then the l-th column in E is given as $\pm(e_i - e_j)$, where the sign depends on the chosen orientation and it is recalled that e_i and e_j are unit vectors. Using this notation, the weighted Laplacian L can be factorized as follows.

Lemma 1: Consider the matrix L in (3) and let E be an oriented incidence matrix of the underlying undirected graph $\mathcal{G}_u$. Then, L can be factored as

$$L = FE^T \tag{9}$$

where $F \in \mathbb{R}^{\bar{n} \times \bar{n}_e}$ has the same structure as E. In particular, let the l-th column of E be given as $e_i - e_j$, then the l-th column of F reads $w_{ij}e_i - w_{ji}e_j$, with w_{ij} the weights in (2).

Proof: The matrix L in (3) can be written as the sum $L = \sum_{(i,j)\in\mathcal{E}_u} L_{ij}$, where L_{ij} characterizes a single edge in $\mathcal{G}_u$. It is readily checked that L_{ij} can be written in the form

$$L_{ij} = (w_{ij}e_i - w_{ji}e_j)(e_i - e_j)^T \tag{10}$$

such that choosing the columns of F and E accordingly leads to (9). ■

The eigenvalues of L can be related to graph-theoretical properties by exploiting the notion of a directed rooted spanning tree (see [16], [30]).

Definition 3: A graph $\mathcal{T}$ is said to be a directed rooted spanning tree if it is a directed tree connecting all vertices of the graph, where every vertex, except the single root vertex, has exactly one incoming directed edge.

The following result can be found in [22], [30].

Lemma 2: Consider the matrix L in (3) with $w_{ij} \geq 0$. Then, L has at least one zero eigenvalue and all nonzero eigenvalues are in the open complex right-half plane. Moreover, L has exactly one zero eigenvalue if and only if the associated graph $\mathcal{G}$ contains a directed rooted spanning tree as a subgraph.

IV. Edge Dynamics and Synchronization

In the remainder of this paper, networked systems $\mathbf{\Sigma}$ will be considered that satisfy the following assumptions.

Assumption 1: The subsystems $\mathbf{\Sigma}_i$ are passive and have the minimal realization (8).

Assumption 2: The interconnection structure characterized by L in (3) is such that the following statements hold:

1) the underlying undirected graph $\mathcal{G}_u$ is a tree (and, thus, $\bar{n}_e = \bar{n} - 1$);
2) the graph $\mathcal{G}$ contains a directed rooted spanning tree as a subgraph.

Here, it is remarked that neither of these items implies the other. Namely, directed graphs satisfying the first item do not necessarily contain a directed rooted spanning tree as a subgraph. Moreover, graphs with cycles might also contain a directed rooted spanning tree as a subgraph, such that the second item does not imply the first.

Remark 1: A relevant example of a networked systems that satisfies Assumptions 1 and 2 is given by large platoons of connected vehicles, see, e.g., [20], [27], [36]. In its simplest form, vehicles with positions s_i are controlled to achieve a desired spacing $\Delta_i = 0$ with $\Delta_i = s_i - s_{i-1}$. Considering vehicles that are controlled to react to their predecessor and follower by adapting their velocity $\dot{s}_i$ as $\dot{s}_i = -\alpha_1\Delta_i + \alpha_2\Delta_{i+1}$, it follows that the platoon dynamics is given as

$$\dot{\Delta}_i = \delta_i, \quad \delta_i = \alpha_1(\Delta_{i-1} - \Delta_i) + \alpha_2(\Delta_{i+1} - \Delta_i) \tag{11}$$

in which the subsystems $\mathbf{\Sigma}_i$ are single integrators satisfying Assumption 1. Moreover, it is clear that the interconnection structure satisfies Assumption 2. ◃

Under Assumption 2, the following lemma can be obtained.

Lemma 3: Let the interconnection structure characterized by L in (3) satisfy Assumption 2 and consider its factorization (9). Then, the matrix $L_e \in \mathbb{R}^{\bar{n}_e \times \bar{n}_e}$ with $\bar{n}_e = \bar{n} - 1$ given as

$$L_e = E^T F \tag{12}$$

has all eigenvalues in the open complex right-half plane and they equal the nonzero eigenvalues of L.

Proof: To prove this lemma, introduce the matrix T as

$$T = \begin{bmatrix} \nu^T \\ E^T \end{bmatrix} \tag{13}$$

where ν is the left eigenvector for the zero eigenvalue of L, i.e., $\nu^T L = 0$. By the second item of Assumption 2 and Lemma 2, this eigenvalue has multiplicity one, such that ν is unique (up to scaling). Also, by exploiting the theory of nonnegative matrices (see, e.g., [17]), it can be shown that all elements of ν are nonnegative (and $\nu \neq 0$). It is therefore assumed that ν is scaled such that $\nu^T \mathbf{1} = 1$. As each column of E has only zero elements except for the pair $(1, -1)$, ν is linearly independent of the columns of E and T as in (13) is nonsingular. Thus, its inverse exists. In particular, it reads

$$T^{-1} = \left[\mathbf{1}\ F(E^T F)^{-1}\right] \tag{14}$$

where it is noted that the inverse of $E^T F$ exists as $E^T F$ is of full rank. Then, the application of the similarity transformation T to L as in (3) leads to

$$TLT^{-1} = \begin{bmatrix} 0 & 0 \\ 0 & E^T F \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & L_e \end{bmatrix}. \tag{15}$$

By Assumption 2 and Lemma 2, L contains only a single zero eigenvalue, which is isolated from the matrix L_e in the representation (15). Consequently, L_e contains all non-zero eigenvalues of L, which are in the open right-half plane by Lemma 2. This proves the desired result. ■

Remark 2: The matrix L_e in (12) is directly related to the dynamics on the edges of the networked system (6). This can be shown using the coordinates $x_e = (E^T \otimes I_n)x$, as will be done in Section V-A. Motivated by this observation, the matrix L_e might be thought of as the (directed and weighted) *edge Laplacian* for the graph $\mathcal{G}$. The edge Laplacian for unweighted and undirected graphs is studied in [39]. ◃

Remark 3: A change in orientation of $\mathcal{G}_u$ can be expressed by a diagonal matrix S with entries in $\{-1, 1\}$, leading to the incidence matrix $\tilde{E} = ES$ and $\tilde{F} = FS$. Then, it is clear that L in (9) is independent of the chosen orientation, but the edge Laplacian L_e satisfies $\tilde{L}_e = \tilde{E}^T \tilde{F} = SL_eS$. However, the results in this paper hold regardless of the choice of orientation. ◃

Next, under the additional assumption that all interconnections are bidirectional (i.e., both $w_{ij} > 0$ and $w_{ji} > 0$ when $(i, j) \in \mathcal{E}_u$), the matrix F in (9) allows for a factorization that will be exploited later.

Lemma 4: Let the interconnection structure characterized by L in (3) satisfy Assumption 2 and consider its factorization (9). Moreover, assume that $w_{ij} > 0$ if and only if $w_{ji} > 0$. Then, there exists diagonal matrices $D_1 \in \mathbb{R}^{\bar{n} \times \bar{n}}$ and $D_2 \in \mathbb{R}^{(\bar{n}-1) \times (\bar{n}-1)}$ satisfying $D_1 \succ 0$ and $D_2 \succ 0$ such that

$$D_1 F D_2 = E. \tag{16}$$

Proof: To prove this lemma, it is recalled that Assumption 2 implies the existence of a directed rooted spanning tree $\mathcal{T}$ as a subgraph of $\mathcal{G}$. Let $\mathcal{E}_{\mathcal{T}}$ denote its set of directed edges, where $(j, i) \in \mathcal{E}_{\mathcal{T}}$ denotes the edge from vertex i to vertex j. The matrices D_1 and D_2 will be explicitly constructed by subsequently considering the $\bar{n} - 1$ edges of $\mathcal{T}$, starting from the root vertex. Thereto, set $k = 0$ and let $\mathcal{V}^k = \{i_0\}$ with i_0 the root vertex. Moreover, the matrices $D_1^0 = I$ and $D_2^0 = I$ are initialized and the matrix $F^0 = D_1^0 F D_2^0$ is defined.

Now, arbitrarily select an edge $(j, i) \in \mathcal{E}_{\mathcal{T}}$ that originates from one of the vertices previously considered and terminates at a vertex not considered before, i.e., that satisfies $i \in \mathcal{V}^k$ and $j \in \mathcal{V} \setminus \mathcal{V}^k$. As $\mathcal{T}$ is a directed rooted spanning tree, such an edge always exists as long as $\mathcal{V}^k \neq \mathcal{V}$. Note that there exists a column in E that equals (up to a possible change of sign) $e_i - e_j$, i.e., that characterizes this edge. Let l be the index of this column. Then, the corresponding column of $F^k := D_1^k F D_2^k$ is given as $((D_1^k)_{ii} w_{ij} e_i - (D_1^k)_{jj} w_{ji} e_j)(D_2^k)_{ll}$. Next, note that the vertex j and the edge l were not considered previously such that the elements $(D_1^k)_{jj}$ and $(D_2^k)_{ll}$, respectively, have not been addressed before. Thus, new matrices D_1^{k+1} and D_2^{k+1} can be defined as copies of D_1^k and D_2^k, with the exception that the elements $(D_1^{k+1})_{jj}$ and $(D_2^{k+1})_{ll}$ are (uniquely) chosen to satisfy

$$\left(\left(D_1^{k+1}\right)_{ii} w_{ij} e_i - \left(D_1^{k+1}\right)_{jj} w_{ji} e_j\right)\left(D_2^{k+1}\right)_{ll} = e_i - e_j. \tag{17}$$

It is easy to see that the solution satisfies $(D_1^{k+1})_{jj} > 0$ and $(D_2^{k+1})_{ll} > 0$. Then, if $\mathcal{V}^{k+1} = \mathcal{V}^k \cup \{j\} \neq \mathcal{V}$, this process is repeated for $k \leftarrow k + 1$.

As $\mathcal{T}$ is a tree, $\mathcal{V}^{k+1}$ equals $\mathcal{V}$ for $k = \bar{n} - 1$ and, in this case, all edges are considered. Moreover, as $\mathcal{T}$ is a directed rooted spanning tree, every vertex in $\mathcal{V}$ can be reached by exactly one unique path from the root vertex, such that no edges in $\mathcal{T}$ (i.e., columns in E and F) are considered twice. The matrices D_1 and D_2 in (16) can now be chosen as $D_1 = D_1^{\bar{n}}$ and $D_2 = D_2^{\bar{n}}$, finalizing the proof of this lemma. ■

The edge Laplacian L_e, introduced in Lemma 3, can be exploited to study synchronization of the networked system $\mathbf{\Sigma}$, as stated in the following theorem from [5].

Theorem 5: Consider the networked system $\mathbf{\Sigma}$ in (6) and let Assumptions 1 and 2 hold. Then, any trajectory of $\mathbf{\Sigma}$ for $u = 0$ satisfies, for all $i, j \in \mathcal{V}$

$$\lim_{t\to\infty} (x_i(t) - x_j(t)) = 0. \tag{18}$$

V. Model Reduction

In this section, a clustering-based model reduction procedure for networked passive systems will be presented, hereby exploiting ideas from balanced truncation. Thereto, edge controllability and edge observability Gramians will be introduced in Section V-A. These Gramians will be exploited to identify the least important edge, whose adjacent vertices are then clustered in a one-step reduction procedure in Section V-B. The repeated application of this procedure leads to a reduction procedure to obtain reduced-order networked systems of arbitrary order, whose properties are analyzed in Section V-C.

A. Edge Controllability and Edge Observability

Balanced truncation, which was introduced in [25], is a successful approach towards model reduction that allows for an insightful interpretation as it is based on quantifying the degree of controllability and observability of a given system (see, e.g., [1], [15]). Moreover, it guarantees stability preservation and provides a computable error bound. Motivated by these properties, ideas from balanced truncation are exploited here for the reduction of networked systems. In particular, the amount of controllability and observability of the dynamics on the edges of the system $\mathbf{\Sigma}$ in (6) will be considered as a basis for subsystem clustering. Thereto, the transformation T in (13) is applied to (6) to obtain the new coordinates $[x_a^T\ x_e^T]^T = (T \otimes I)x$. Then, it can be concluded from (15) that the dynamics of x_a and x_e are independent.

Specifically, the coordinate $x_a \in \mathbb{R}^n$ provides a (weighted) average of the states of the individual subsystems and satisfies the dynamics

$$\mathbf{\Sigma}_a : \begin{cases} \dot{x}_a = Ax_a + (\nu^T G \otimes B)u, \\ y_a = (H\mathbf{1} \otimes C)x_a. \end{cases} \tag{19}$$

Therefore, $\mathbf{\Sigma}_a$ will be referred to as the *average system*. Next, the state $x_e \in \mathbb{R}^{(\bar{n}-1)n}$ describes, for each edge, the difference of the states of the adjacent subsystems (see also Remark 2). It can be shown (by noting that $E^T L = L_e E^T$) that x_e satisfies the dynamics

$$\mathbf{\Sigma}_e : \begin{cases} \dot{x}_e = (I \otimes A - L_e \otimes BC)x_e + (G_e \otimes B)u \\ y_e = (H_e \otimes C)x_e \end{cases} \tag{20}$$

with $G_e = E^T G$ and $H_e = HF(E^T F)^{-1}$. Here, it is noted that the definition of H_e is the result of the expression of the inverse transformation (14). The system $\mathbf{\Sigma}_e$ in (20) will be called the *edge system* and is asymptotically stable by Assumptions 1 and 2 and Theorem 5, where it is noted that synchronization of $\mathbf{\Sigma}$ is equivalent to asymptotic stability of the associated edge system $\mathbf{\Sigma}_e$.

The controllability properties of the edge system characterize whether adjacent subsystems in the networked system $\mathbf{\Sigma}$ in (6) can be controlled independently, motivating the following definition (note that $G_e = E^T G$).

Definition 4: A matrix $\tilde{P}_e = \tilde{\Pi}^c \otimes Q^{-1}$ is said to be a generalized edge controllability Gramian for the networked system $\mathbf{\Sigma}$ satisfying Assumptions 1 and 2 if the matrix $\tilde{\Pi}^c \in \mathbb{R}^{(\bar{n}-1)\times(\bar{n}-1)}$ such that $\tilde{\Pi}^c \succcurlyeq 0$ is diagonal and satisfies

$$L_e \tilde{\Pi}^c + \tilde{\Pi}^c L_e^T - E^T G G^T E \succcurlyeq 0. \tag{21}$$

Similar to the characterization of edge controllability in Definition 4, edge *observability* can be characterized. However, the latter characterization is obtained on the basis of a different realization of the edge system (20), which will be shown in Section V-B (see in particular Remark 6) to have desirable properties when model reduction by clustering is performed. In particular, coordinates $x_f = ((E^T F)^{-1} \otimes I)x_e \in \mathbb{R}^{(\bar{n}-1)n}$ are considered, such that the transformation of (20) leads to

$$\mathbf{\Sigma}_f : \begin{cases} \dot{x}_f = (I \otimes A - L_e \otimes BC)x_f + (G_f \otimes B)u \\ y_e = (H_f \otimes C)x_f \end{cases} \tag{22}$$

with $G_f = (E^T F)^{-1} E^T G$ and $H_f = HF$. Clearly, $\mathbf{\Sigma}_f$ is asymptotically stable and its input-output behaviour equals that of $\mathbf{\Sigma}_e$ in (20). Nonetheless, for clarity of exposition, the system $\mathbf{\Sigma}_f$ will be referred to as the *dual edge system*.

Then, the observability properties of the dual edge system provide a characterization of whether adjacent subsystems in the networked system $\mathbf{\Sigma}$ in (6) can be distinguished, leading to the following definition and where it is recalled that $H_f = HF$.

Definition 5: A matrix $\tilde{Q}_f = \tilde{\Pi}^o \otimes Q$ is said to be a generalized edge observability Gramian for the networked system $\mathbf{\Sigma}$ satisfying Assumptions 1 and 2 if the matrix $\tilde{\Pi}^o \in \mathbb{R}^{(\bar{n}-1)\times(\bar{n}-1)}$ such that $\tilde{\Pi}^o \succcurlyeq 0$ is diagonal and satisfies

$$L_e^T \tilde{\Pi}^o + \tilde{\Pi}^o L_e - F^T H^T H F \succcurlyeq 0. \tag{23}$$

The existence of the matrices $\tilde{\Pi}^c$ and $\tilde{\Pi}^o$ characterizing the generalized edge controllability and observability Gramians in (21) and (23), respectively, is guaranteed when all interconnected vertices are bidirectionally coupled. This is formalized in the following lemma.

Lemma 6: Let the interconnection structure characterized by L in (3) satisfy Assumption 2 and assume that $w_{ij} > 0$ if and only if $w_{ji} > 0$. Then, there exist diagonal matrices $\tilde{\Pi}^c \succcurlyeq 0$ and $\tilde{\Pi}^o \succcurlyeq 0$ satisfying (21) and (23), respectively.

Proof: The theorem will be proven for the case of controllability. The result on observability follows similarly.

To prove existence of a diagonal solution $\tilde{\Pi}^c$ to (21), it is remarked that the statements in Lemma 4 hold and the matrix F in (9) can be written as $F = D_1^{-1} E D_2^{-1}$. Then

$$D_2^{-\frac{1}{2}} L_e D_2^{\frac{1}{2}} = D_2^{-\frac{1}{2}} E^T F D_2^{\frac{1}{2}} = D_2^{-\frac{1}{2}} E^T D_1^{-1} E D_2^{-\frac{1}{2}} \tag{24}$$

is a symmetric matrix with the same eigenvalues as L_e, which are all positive by Assumption 2 and Lemma 3. As a result, also the matrix $E^T D_1^{-1} E$ is symmetric and positive definite. By exploiting this, it follows that:

$$\begin{aligned} L_e D_2 + D_2 L_e^T &= E^T F D_2 + D_2 F^T E \\ &= 2E^T D_1^{-1} E \succ 0. \end{aligned} \tag{25}$$

Thus, as the left-hand side of (25) is positive definite, there exists a parameter ε such that

$$L_e D_2 + D_2 L_e^T \succ \varepsilon E^T G G^T E \tag{26}$$

showing that $\tilde{\Pi}^c = \varepsilon^{-1} D_2$ is a (positive definite) diagonal solution to (21). Consequently, the generalized edge controllability Gramian $\tilde{P}_e$ as in Definition 4 exists. ■

Lemma 6 thus guarantees the existence of the generalized edge Gramians in Definitions 4 and 5. However, the main motivation for the introduction of these generalized Gramians follows from the following theorem.

Theorem 7: Consider the networked system $\mathbf{\Sigma}$ in (6) satisfying Assumptions 1 and 2. Moreover, assume that matrices $\tilde{\Pi}^c \succcurlyeq 0$ and $\tilde{\Pi}^o \succcurlyeq 0$ satisfying (21) and (23), respectively, exist. Then, the generalized edge controllability Gramian $\tilde{P}_e$ and the generalized edge observability Gramian $\tilde{Q}_f$ satisfy

$$\bar{P}_e \preccurlyeq \tilde{P}_e = \tilde{\Pi}^c \otimes Q^{-1} \tag{27}$$

$$\bar{Q}_f \preccurlyeq \tilde{Q}_f = \tilde{\Pi}^o \otimes Q \tag{28}$$

with $\bar{P}_e$ the controllability Gramian of $\mathbf{\Sigma}_e$ in (20) and $\bar{Q}_f$ the observability Gramian of $\mathbf{\Sigma}_f$ in (22).

Proof: The controllability part can be proven by showing that the matrix

$$\begin{aligned} \Delta := &(I \otimes A - L_e \otimes BC)(\tilde{\Pi}^c \otimes Q^{-1}) \\ &+ (\tilde{\Pi}^c \otimes Q^{-1})(I \otimes A - L_e \otimes BC)^T \\ &+ (G_e \otimes B)(G_e \otimes B)^T \end{aligned} \tag{29}$$

is negative definite, hereby using results from [6] (see [19] for a similar result). The observability case can be proven similarly. ■

By Theorem 7, the generalized edge Gramians thus provide upper bounds for the actual Gramians of the edge system (20) and dual edge system (22). Here, the Kronecker product structure of the generalized edge Gramians amounts to a decomposition in two parts, where the first part is only dependent on the properties of the interconnection topology as characterized through the edge Laplacian L_e [through (21) and (23)]. The second part is related to the individual subsystems only, through the storage function that characterizes their passivity property. This decomposition thus allows for the interpretation of controllability properties on the basis of the interconnection topology only, through the matrices $\tilde{\Pi}^c$ and $\tilde{\Pi}^o$ given by (21) and (23), respectively. This is highly beneficial in the development of a reduction method based on subsystem clustering, as this reduction is essentially performed on the interconnection level.

In particular, the diagonal structure of the matrices $\tilde{\Pi}^c$ and $\tilde{\Pi}^o$ allows for a suitable interpretation of the controllability and observability properties of individual edges. Namely, by Theorem 7, $\tilde{\Pi}^c \otimes Q^{-1}$ provides an upper bound on the edge controllability Gramian, and therefore a lower bound on the energy required to influence each individual edge (see, e.g., [1], [15] for an energy interpretation of the controllability and observability Gramians). Here, it is remarked that, if an edge is hard to control, it implies that the adjacent subsystems are hard to influence independently and will thus show similar input-to-state behavior. Similarly, if an edge is hard to observe (as characterized through $\tilde{\Pi}^o \otimes Q$), the two adjacent subsystems are hard to distinguish. These observations will provide the main motivation for clustering the vertices adjacent to such edges.

Remark 4: Clearly, the matrices $\tilde{\Pi}^c$ and $\tilde{\Pi}^o$ in Definitions 4 and 5, which in the remainder of this paper will also (abusively) be referred to as the generalized edge Gramians, are not unique. However, it might be expected that a better reduced-order networked system can be obtained when the generalized edge Gramians provide a tight upper bound on the regular edge Gramians Π^c and Π^o. A good heuristic to achieve this is the minimization of the trace of $\tilde{\Pi}^c$ and $\tilde{\Pi}^o$ under the constraints (21) and (23) (see, e.g., [4]), which can be achieved by standard tools for solving linear matrix inequalities. The intuition that the minimization of the diagonal values of the generalized edge Gramians leads to better reduced-order models will be confirmed by error analysis in Section V-C. ◁

B. One-Step Model Reduction Through Clustering

The generalized edge controllability and generalized edge observability Gramians introduced in Section V-A provide a characterization of the degree of controllability and observability of each individual edge. The combination of these aspects provides a measure of the contribution of each edge on the input-output behavior of the networked system $\mathbf{\Sigma}$, such that the least important edge can be identified. Roughly speaking, the least important edge has both a low degree of controllability and a low degree of observability. However, details on how to select this edge on the basis of the generalized edge Gramians will be given in Section V-C, where the most suitable approach will be identified by means of error analysis.

In this section, it is assumed that the least important edge has been identified and a one-step reduction (i.e., involving only this single edge) will be considered. As motivated earlier, the subsystems adjacent to this edge are hard to control individually and hard to distinguish and it is natural to cluster these two subsystems. Here, for ease of notation, it is assumed that the least important vertices are given by i and j, where $i = \bar{n} - 1$ and $j = \bar{n}$. This can always be achieved by a suitable permutation of the vertex numbers. Then, the projection matrices $V \in \mathbb{R}^{\bar{n} \times (\bar{n}-1)}$ and $W \in \mathbb{R}^{\bar{n} \times (\bar{n}-1)}$ are introduced as

$$V = \begin{bmatrix} I & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}, \quad W = \begin{bmatrix} I & 0 \\ 0 & \frac{w_{ji}}{w_{ij}+w_{ji}} \\ 0 & \frac{w_{ij}}{w_{ij}+w_{ji}} \end{bmatrix} \tag{30}$$

clearly characterizing a cluster of the final two vertices. Here, the elements in the bottom right corner of W are directly related to the weights of the (directed) edges between vertices i and j.

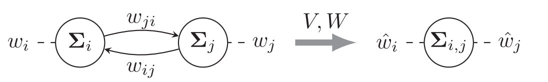

Fig. 1. Schematic illustration of the clustering procedure, where projection with V, W as in (30) amounts to the clustering of subsystems $\boldsymbol{\Sigma}_i$ and $\boldsymbol{\Sigma}_j$ in one new subsystem $\boldsymbol{\Sigma}_{i,j}$. Herein, the weights of the links connecting the cluster $\boldsymbol{\Sigma}_{i,j}$ to the rest of the network get updated.

The matrices (30) are used to define the projection $(V \otimes I)(W \otimes I)^T$, which is biorthogonal (i.e., a Petrov-Galerkin projection, see [1]) due to the property $W^T V = I$. The application of this projection to the system matrix in the networked system $\boldsymbol{\Sigma}$ in (6) leads to

$$(W \otimes I)^T (I_{\bar{n}} \otimes A - L \otimes BC)(V \otimes I) = W^T V \otimes A - W^T L V \otimes BC \quad (31)$$

with $W^T V = I_{\bar{n}-1}$ and $W^T L V \in \mathbb{R}^{(\bar{n}-1)\times(\bar{n}-1)}$, as follows from the properties of the Kronecker product (see [7]).

By projecting also the input and output matrices of (6), the one-step reduced order system

$$\hat{\boldsymbol{\Sigma}}_{\bar{n}-1} : \begin{cases} \dot{\xi} = (I \otimes A - W^T L V \otimes BC)\xi + (W^T G \otimes B)u \\ \hat{y} = (HV \otimes C)\xi \end{cases} \quad (32)$$

is obtained, where $\xi \in \mathbb{R}^{(\bar{n}-1)n}$ provides an approximation of x as $x \approx (V \otimes I)\xi$. It is clear from (32) that the dynamics of the subsystems is not influenced by this reduction. Instead, only the interconnection structure has been affected to introduce the cluster of the vertices i and j, as illustrated in Fig. 1.

In order to analyze the properties of the reduced-order networked system (32), the edge connecting vertices i and j (i.e., the least important edge earlier identified) is assumed to be characterized by the l-th column of the original incidence matrix E (and weighted incidence matrix F), where $l = \bar{n}_e = \bar{n} - 1$. As for the vertices, this can always be achieved by a suitable permutation of the edge numbers. Now, the matrices E and F in (9) can be partitioned accordingly, leading to

$$E = \begin{bmatrix} E_{00} & 0 \\ E_{i0} & E_{il} \\ E_{j0} & E_{jl} \end{bmatrix}, \qquad F = \begin{bmatrix} F_{00} & 0 \\ F_{i0} & F_{il} \\ F_{j0} & F_{jl} \end{bmatrix}. \quad (33)$$

Here, it is noted that the zero entries in both E and F result from the fact that the corresponding column represents the edge connecting vertex i to j. Specifically, $E_{il} \in \{-1, 1\}$ and $E_{jl} = -E_{il}$. Similarly, $F_{il} = w_{ij} E_{il}$ and $F_{jl} = w_{ji} E_{jl}$, as follows from Lemma 1.

Using the partitioning (33), the matrix $\hat{L} = W^T L V$ characterizing the interconnection topology of the reduced-order networked system $\hat{\boldsymbol{\Sigma}}_{\bar{n}-1}$ in (32) can be shown to satisfy the following properties.

Lemma 8: Let the interconnection structure characterized by L in (3) satisfy Assumption 2 and consider its factorization (9), in which the factors E and F are partitioned as in (33). Moreover, let $\hat{L} = W^T L V$ be the reduced-order interconnection matrix obtained by projection using the matrices (30), let $\hat{\mathcal{G}}$ be the corresponding graph on $\bar{n} - 1$ vertices and $\hat{\mathcal{G}}_u$ the underlying undirected graph. Then, the following results hold:

1) The matrix $\hat{L}$ can be factored as $\hat{L} = \hat{F}\hat{E}^T$, where $\hat{E} \in \mathbb{R}^{(\bar{n}-1)\times(\bar{n}-2)}$ and $\hat{F} \in \mathbb{R}^{(\bar{n}-1)\times(\bar{n}-2)}$ are given by

$$\hat{E} = \begin{bmatrix} E_{00} \\ E_{i0} + E_{j0} \end{bmatrix} \quad (34)$$

$$\hat{F} = \begin{bmatrix} F_{00} \\ \frac{w_{ji}}{w_{ij}+w_{ji}} F_{i0} + \frac{w_{ij}}{w_{ij}+w_{ji}} F_{j0} \end{bmatrix}. \quad (35)$$

2) The underlying undirected graph $\hat{\mathcal{G}}_u$ is a tree;
3) The graph $\hat{\mathcal{G}}$ contains a directed rooted spanning tree as a subgraph.

Proof: The first item can be proven by exploiting the factorization of L as in (9), from which it follows that

$$W^T L V = W^T F E^T V = (W^T F)(V^T E)^T. \quad (36)$$

The computation of $V^T E$, hereby using (33), leads to

$$V^T E = \begin{bmatrix} E_{00} & 0 \\ E_{i0} + E_{j0} & 0 \end{bmatrix} \quad (37)$$

where it is noted that the final column contains all zeros since $E_{il} + E_{jl} = 0$. Namely, the l-th (with $l = \bar{n} - 1$) column of E characterizes the edge that connects vertices i and j, such that $E_{il} \in \{1, -1\}$ and $E_{jl} = -E_{il}$. Similarly

$$W^T F = \begin{bmatrix} F_{00} & 0 \\ \frac{w_{ji}}{w_{ij}+w_{ji}} F_{i0} + \frac{w_{ij}}{w_{ij}+w_{ji}} F_{j0} & 0 \end{bmatrix} \quad (38)$$

hereby using a similar argument to prove that the final column contains all zeros. Herein, the choice of the weights in W as in (30) is crucial. Also, it is noted that the bottom row in (38) specifies how the weights of the links that connect the cluster to the rest of the network are updated (see Fig. 1). Finally, setting $\hat{E}$ and $\hat{F}$ as the nonzero columns of $V^T E$ and $W^T F$, respectively, proves the first item in the statement of the theorem.

The second and third item of the proof are very intuitive, but a formal proof is given in [5]. ■

As the selection of suitable vertices for clustering is based on properties of the edge system (20) and dual edge system (22), it is natural to analyze the reduced-order system in similar coordinates. Therefore, the transformation matrix $\hat{T} = [\hat{\nu} \ \hat{E}]^T$ is introduced, which is of the same form as (13). Here, $\hat{\nu}$ is the left eigenvector for the zero eigenvalue of $\hat{L}$, thus satisfying $\hat{\nu}^T \hat{L} = 0$ and $\hat{\nu}^T \mathbf{1} = 1$. Moreover, $\hat{E}$ is the reduced-order incidence matrix of the underlying undirected graph $\hat{\mathcal{G}}_u$ as in Lemma 8 [see (34)].

Then, by performing the coordinate transformation $[\xi_a^T \ \xi_e^T]^T = (\hat{T} \otimes I)\xi$, it follows that the dynamics of the reduced-order average system $\hat{\boldsymbol{\Sigma}}_a$ is given as

$$\hat{\boldsymbol{\Sigma}}_a : \begin{cases} \dot{\xi}_a = A\xi_a + (\hat{\nu}^T W^T G \otimes B)u \\ \hat{y}_a = (HV\mathbf{1} \otimes C)\xi_a. \end{cases} \quad (39)$$

Here, it is noted that $\xi_a \in \mathbb{R}^n$, such that its order equals that of $\mathbf{\Sigma}_a$ in (19). Nonetheless, $\hat{\mathbf{\Sigma}}_a$ will be referred to as the reduced-order average system.

Similarly to the edge system $\mathbf{\Sigma}_e$ in (20), the (one-step) reduced-order edge system $\hat{\mathbf{\Sigma}}_e$ is given as

$$\hat{\mathbf{\Sigma}}_e : \begin{cases} \dot{\xi}_e = (I \otimes A - \hat{L}_e \otimes BC)\xi_e + (\hat{G}_e \otimes B)u \\ \hat{y}_e = (\hat{H}_e \otimes C)\xi_e \end{cases} \tag{40}$$

where $\xi_e \in \mathbb{R}^{(\bar{n}-2)\bar{n}}$ and $\hat{L}_e = \hat{E}^T\hat{F}$ is the reduced-order edge Laplacian for the graph $\hat{\mathcal{G}}$. In (40), the matrices $\hat{G}_e$ and $\hat{H}_e$ are given as $\hat{G}_e = \hat{E}^T W^T G$ and $\hat{H}_e = HV\hat{F}(\hat{E}^T\hat{F})^{-1}$, respectively. Moreover, after expressing (40) in new coordinates $\xi_f = ((\hat{E}^T\hat{F})^{-1} \otimes I)\xi_e$, the reduced-order dual edge system $\hat{\mathbf{\Sigma}}_f$ is obtained. Similar to the high-order counterpart in (22), it has the same form as the reduced-order edge system (40) with new external input matrix $\hat{G}_f = (\hat{E}^T\hat{F})^{-1}\hat{E}^T W^T G$ and external output matrix $\hat{H}_f = HV\hat{F}$.

It is recalled that the selection of the least important edge is performed on the basis of the generalized edge Gramians in Definitions 4 and 5, which are in turn calculated through the edge dynamics (20) and dual edge dynamics (22). After expressing the generalized edge Gramians $\tilde{\Pi}^c$ and $\tilde{\Pi}^o$ as

$$\tilde{\Pi}^c = \operatorname{diag}\left\{\pi_1^c, \pi_2^c, \dots \pi_{\bar{n}-1}^c\right\} \tag{41}$$

$$\tilde{\Pi}^o = \operatorname{diag}\left\{\pi_1^o, \pi_2^o, \dots \pi_{\bar{n}-1}^o\right\} \tag{42}$$

these Gramians can be related to the one-step reduced-order networked system $\hat{\mathbf{\Sigma}}_{\bar{n}-1}$ in (32) through the edge system $\hat{\mathbf{\Sigma}}_e$ in (40) and the reduced-order dual edge system $\hat{\mathbf{\Sigma}}_f$. This is formalized as follows.

Theorem 9: Consider the networked system $\mathbf{\Sigma}$ in (6) satisfying Assumptions 1 and 2 and the reduced-order networked system $\hat{\mathbf{\Sigma}}_{\bar{n}-1}$ in (32). Let the generalized edge controllability Gramain $\tilde{\Pi}^c$ and generalized edge observability Gramian $\tilde{\Pi}^o$ be written as (41) and (42), respectively. Then,

1) $\tilde{\Pi}_1^c := \operatorname{diag}\{\pi_1^c, \dots \pi_{\bar{n}-2}^c\}$ is a generalized edge controllability Gramian for $\hat{\mathbf{\Sigma}}_{\bar{n}-1}$;
2) $\tilde{\Pi}_1^o := \operatorname{diag}\{\pi_1^o, \dots \pi_{\bar{n}-2}^o\}$ is a generalized edge observability Gramian for $\hat{\mathbf{\Sigma}}_{\bar{n}-1}$.

Proof: In order to prove the theorem, it will be shown first that the reduced-order (dual) edge system $\hat{\mathbf{\Sigma}}_e$ ($\hat{\mathbf{\Sigma}}_f$) can directly be obtained from its original (i.e., non-reduced) counterpart $\mathbf{\Sigma}_e$ ($\mathbf{\Sigma}_f$). Thereto, the matrices in (20) and (22) are partitioned as

$$L_e = \begin{bmatrix} L_{e,11} & L_{e,12} \\ L_{e,21} & L_{e,22} \end{bmatrix}, \; G_e = \begin{bmatrix} G_{e,1} \\ G_{e,2} \end{bmatrix}$$
$$H_f = \begin{bmatrix} H_{f,1} & H_{f,2} \end{bmatrix} \tag{43}$$

after which it can be checked [by using the partitioning (33) and the relations (34), (35)] that

$$\hat{L}_e = L_{e,11} - L_{e,12}L_{e,22}^{-1}L_{e,21} \tag{44}$$

$$\hat{G}_e = G_{e,1} - L_{e,12}L_{e,22}^{-1}G_{e,2} \tag{45}$$

$$\hat{H}_f = H_{f,1} - H_{f,2}L_{e,22}^{-1}L_{e,21}. \tag{46}$$

Next, a result from [13] can be used to finalize the proof. Namely, the application of a projection matrix $T_c = [I \;\; -L_{e,12}L_{e,22}^{-1}]$ to (21) yields

$$T_c\left(L_e\tilde{\Pi}^c + \tilde{\Pi}^c L_e^T - G_e G_e^T\right)T_c^T$$
$$= \hat{L}_e\tilde{\Pi}_1^c + \tilde{\Pi}_1^c\hat{L}_e^T - \hat{G}_e\hat{G}_e^T \succcurlyeq 0 \tag{47}$$

where the partitioning (43) is used as well as (44) and (45). It can be seen that the right-hand side of the equality in (47) characterizes a generalized edge controllability Gramian for the reduced-order system $\hat{\mathbf{\Sigma}}$, proving the first item of the theorem. The second item can be proven similarly. ■

Theorem 9 shows that the relevant blocks of the generalized edge Gramians are preserved under reduction. This is important, as it will be shown in Section V-C that this facilitates the repetitive application of the one-step reduction procedure discussed in this section, yielding reduced-order networked systems of arbitrary dimension.

Remark 5: The relation (44) represents a Schur complement of the edge Laplacian L_e. The Schur complement of a Laplacian matrix also forms the basis of so-called Kron reduction of graphs, which finds application in the scope of reduction of electrical networks (see [11]). ◁

Remark 6: The fact that Theorem 9 gives similar statements for the generalized edge controllability and generalized edge observability Gramians is directly dependent on the choice to consider edge controllability and edge observability in different and specific coordinates, thus providing a motivation for this choice. Here, it is recalled that the edge system $\mathbf{\Sigma}_e$ in (20) is used to study controllability, whereas the dual edge system $\mathbf{\Sigma}_f$ in (22) is exploited for the observability case. The evaluation of controllability and observability in the same coordinates would not lead to a result as in Theorem 9 for both Gramians simultaneously, hereby prohibiting the repeated application of one-step reductions through clustering. Moreover, this choice would not allow for the computation of a bound on the reduction error in Section V-C. Consequently, the use of different coordinates for the expression of the generalized edge Gramians is crucial for the development of the clustering-based reduction technique in this paper and the parameters π_i^c and π_i^o in (41), (42) play a similar role as the Hankel singular values in balanced truncation (see [15] for the latter). ◁

C. Preservation of Synchronization and Error Bound

The one-step reduction procedure through the projection using the matrices (30) will be repeatedly applied to obtain reduced-order networked systems of arbitrary order. However, before treating this in more detail, the quality of the approximation obtained by a one-step reduction will be analyzed.

To this end, the output error introduced by reduction is considered. Herein, it is recalled that the output y of the networked system $\mathbf{\Sigma}$ in (6) can be found as the sum of the output of the average system $\mathbf{\Sigma}_a$ in (19) and the edge system $\mathbf{\Sigma}_e$ in (20), i.e., $y = y_a + y_e$. A similar statement holds for the reduced-order networked system $\hat{\mathbf{\Sigma}}_{\bar{n}-1}$ in (32), such that $\hat{y} = \hat{y}_a + \hat{y}_e$. Error analysis for the networked system will therefore be considered for the average system and edge system separately. First, the reduced-order average system $\hat{\boldsymbol{\Sigma}}_a$ equals $\boldsymbol{\Sigma}_a$, as formalized as follows.

Theorem 10: Consider the networked system $\boldsymbol{\Sigma}$ in (6) satisfying Assumptions 1 and 2 and let $\hat{\boldsymbol{\Sigma}}_{\bar{n}-1}$ in (32) be a one-step approximation obtained by projection. Let $\boldsymbol{\Sigma}_a$ in (19) and $\hat{\boldsymbol{\Sigma}}_a$ in (39) be the corresponding average systems. Then, for any input function $u(\cdot)$ and initial conditions satisfying $x_a(0) = \xi_a(0)$, the outputs satisfy $y_a(t) = \hat{y}_a(t)$ for all $t \geq 0$.

Proof: To prove the theorem, the error coordinates e_a are introduced as $e_a = x_a - \xi_a$. Next, from the definition of V in (30) it is easily seen that $V\mathbf{1}_{\bar{n}-1} = \mathbf{1}_{\bar{n}}$, such that it follows from (19) and (39) that the output error can be written as $y_a - \hat{y}_a = (H\mathbf{1} \otimes C)e_a$. The dynamics of the error can be obtained from (19) and (39) and reads

$$\dot{e}_a = Ae_a + \left((\nu^T - \hat{\nu}^T W^T)G \otimes B\right) u. \tag{48}$$

In the remainder of the proof, it will be shown that $\nu = W\hat{\nu}$, such that (48) is independent of the input u, proving the theorem.

It is recalled that ν and $\hat{\nu}$ represent the left eigenvector for the single zero eigenvalue of L and $\hat{L}$, respectively, i.e., $\nu^T L = 0$ and $\hat{\nu}^T \hat{L} = 0$. By the factorizations of L as in (9) and $\hat{L} = \hat{F}\hat{E}^T$ (see Lemma 8) this is equivalent to $F^T\nu = 0$ and $\hat{F}^T\hat{\nu} = 0$ such that ν and $\hat{\nu}$ form a basis for the null spaces (which are guaranteed to be of dimension one by the factorizations of L_e and $\hat{L}_e$, where the latter are full-rank) of F^T and $\hat{F}^T$. Then, after recalling the derivation of $\hat{F}$ in (38), it can be concluded that $\hat{\nu}^T W^T F = \hat{\nu}^T[\hat{F}\ 0] = 0$ and the vector $W\hat{\nu}$ is in the null space of F^T. Consequently, there exists a scalar c such that $cW\hat{\nu} = \nu$. Pre-multiplication with $\mathbf{1}_{\bar{n}}^T$ yields $c\mathbf{1}_{\bar{n}}^T W\hat{\nu} = \mathbf{1}_{\bar{n}}^T\nu$. Then, by recalling the definition of W in (30), it can be seen that $\mathbf{1}_{\bar{n}}^T W = \mathbf{1}_{\bar{n}-1}^T$. Now, by exploiting that ν and $\hat{\nu}$ have nonnegative elements and the standing assumption that they are scaled such that $\mathbf{1}_{\bar{n}}^T\nu = 1$ and $\mathbf{1}_{\bar{n}-1}^T\hat{\nu} = 1$, it can be seen that $c = 1$ and the result follows. ■

Next, the error between the high-order edge system $\boldsymbol{\Sigma}_e$ and the reduced-order edge system $\hat{\boldsymbol{\Sigma}}_e$ is considered. Thereto, the matrix M is introduced and partitioned according to (43) as

$$M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} := L_e^{-1} \tag{49}$$

such that M_{22} is scalar. Then, when all interconnected subsystems are bidirectionally coupled, the error bound on the edge systems is given as follows.

Theorem 11: Consider the networked system $\boldsymbol{\Sigma}$ in (6) satisfying Assumptions 1 and 2, where $w_{ij} > 0$ if and only if $w_{ji} > 0$. Let $\hat{\boldsymbol{\Sigma}}_{\bar{n}-1}$ in (32) be a one-step approximation obtained by projection. Let $\boldsymbol{\Sigma}_e$ in (20) and $\hat{\boldsymbol{\Sigma}}_e$ in (40) be the corresponding edge systems. Then, for any input function $u(\cdot) \in \mathcal{L}_2^{\bar{m}}([0,\infty))$ and initial conditions $x_e(0) = 0$, $\xi_e(0) = 0$, the outputs satisfy

$$\|y_e - \hat{y}_e\|_2 \leq 2M_{22}\left(\tilde{\Pi}_2^c\tilde{\Pi}_2^o\right)^{\frac{1}{2}}\|u\|_2 \tag{50}$$

with M_{22} as in (49) satisfying $M_{22} > 0$. Moreover, $\tilde{\Pi}_2^c := \pi_{\bar{n}-1}^c$ and $\tilde{\Pi}_2^o := \pi_{\bar{n}-1}^o$ are the discarded parts of the generalized edge controllability and generalized edge observability Gramians as in (41) and (42).

Proof: The proof can be found in Appendix A. ■

In the results discussed so far, a one-step reduction was considered, in which two neighbouring vertices are clustered to obtain a reduced-order networked system. In order to define the reduction of networked systems to an arbitrary order, it is recalled that the diagonal generalized edge controllability Gramian $\tilde{\Pi}^c$ and generalized edge observability Gramian $\tilde{\Pi}^o$ in Definitions 4 and 5, respectively, are written as (41) and (42). Moreover, it is assumed that the edges are numbered such that the inequalities

$$(M)_{11}^2\pi_1^c\pi_1^o \geq \ldots \geq (M)_{(\bar{n}-1)(\bar{n}-1)}^2\pi_{\bar{n}-1}^c\pi_{\bar{n}-1}^o \geq 0 \tag{51}$$

hold, where $(M)_{ii}$ is the i-th element on the diagonal of $M = L_e^{-1}$. The ordering (51) is motivated by the one-step error bound (50), which indicates that small values of the products in (51) lead to smaller error bounds and potentially better reduced-order systems.

Remark 7: The ordering (51) can always be obtained by a suitable permutation of the edges in the original coordinates (20). After introducing the permutation matrix S and new coordinates $x_e = (S \otimes I)\tilde{x}_e$, the edge Laplacian L_e is transformed to $S^T L_e S$ (consequently, M becomes $S^T MS$), whereas the generalized edge Gramians $\tilde{\Pi}^c$ and $\tilde{\Pi}^o$ can be written as $S^T\tilde{\Pi}^c S$ and $S^T\tilde{\Pi}^o S$, respectively, after transformation. ◁

When the edges are numbered such that the inequalities (51) hold, a reduced-order networked system consisting of $\bar{k}$ subsystems can be obtained by introducing

$$\begin{aligned} \bar{V} &= S_{\bar{n}-1}V_{\bar{n}-1}S_{\bar{n}-2}V_{\bar{n}-2}\cdots S_{\bar{k}}V_{\bar{k}} \\ \bar{W} &= S_{\bar{n}-1}W_{\bar{n}-1}S_{\bar{n}-2}W_{\bar{n}-2}\cdots S_{\bar{k}}W_{\bar{k}} \end{aligned} \tag{52}$$

where the projection matrices V_l and W_l are defined as

$$V_l = \begin{bmatrix} I_{l-1} & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}, \quad W_l = \begin{bmatrix} I_{l-1} & 0 \\ 0 & \frac{w_{j_l i_l}}{w_{i_l j_l} + w_{j_l i_l}} \\ 0 & \frac{w_{i_l j_l}}{w_{i_l j_l} + w_{j_l i_l}} \end{bmatrix}. \tag{53}$$

The projection (52) can be considered as a series of one-step projections $S_l V_l$ (and $S_l W_l$). Here, S_l is a permutation matrix that renumbers the vertices (subsystems) such that the vertices i_l and j_l are renumbered as l and $l-1$, with i_l and j_l the vertices adjacent to the edge l that is removed in this step. Then, V_l and W_l are of the form (30) and ensure that the relevant vertices are clustered.

The application of the projection given by (52) to the original networked system $\boldsymbol{\Sigma}$ (with $\bar{n}$ subsystems) in (6) leads to a reduced-order networked system with $\bar{k}$ subsystems $\hat{\boldsymbol{\Sigma}}_{\bar{k}}$ as

$$\hat{\boldsymbol{\Sigma}}_{\bar{k}} : \begin{cases} \dot{\xi} = (I \otimes A - \bar{W}^T L\bar{V} \otimes BC)\xi + (\bar{W}^T G \otimes B)u \\ \hat{y} = (H\bar{V} \otimes C)\xi \end{cases} \tag{54}$$

with $\xi \in \mathbb{R}^{\bar{k}n}$ the reduced-order state vector.

Then, the results on properties of the reduced-order network system obtained by a one-step reduction can be extended to obtain results on $\hat{\boldsymbol{\Sigma}}_{\bar{k}}$, as formalized in the following theorem.

Theorem 12: Consider the networked system $\mathbf{\Sigma}$ in (6) satisfying Assumptions 1 and 2 and let $\hat{\mathbf{\Sigma}}_{\bar{k}}$ be a reduced-order networked system obtained by the series of one-step clusterings (52). Let $\hat{\mathcal{G}}_{\bar{k}}$ be the corresponding graph on $\bar{k}$ vertices and $\hat{\mathcal{G}}_{u,\bar{k}}$ the underlying undirected graph. Then, $\hat{\mathcal{G}}_{u,\bar{k}}$ is a tree and $\hat{\mathcal{G}}_{\bar{k}}$ contains a directed rooted spanning tree as a subgraph. Moreover, any trajectory of $\hat{\mathbf{\Sigma}}_{\bar{k}}$ satisfies, for all $i, j \in \hat{\mathcal{V}} := \{1, 2, \ldots, \bar{k}\}$

$$\lim_{t\to\infty} (\xi_i(t) - \xi_j(t)) = 0. \tag{55}$$

If, in addition, $w_{ij} > 0$ if and only if $w_{ji} > 0$, then the output error $y - \hat{y}$ satisfies

$$\|\hat{y} - y\|_2 \leq 2\left(\sum_{l=k}^{\bar{n}-1} (M)_{ll}\sqrt{\pi_l^c \pi_l^o}\right)\|u\|_2 \tag{56}$$

for all trajectories $x(\cdot)$ of $\mathbf{\Sigma}$ and $\xi(\cdot)$ of $\hat{\mathbf{\Sigma}}_{\bar{k}}$ such that $x(0) = 0$ and $\xi(0) = 0$ and for any input function $u(\cdot) \in \mathcal{L}_2^{\bar{m}}([0, \infty))$.

Proof: To prove the theorem, it is first noted that Lemma 8 guarantees that the interconnection structure obtained by a one-step reduction satisfies the conditions in Assumption 2. The repeated application of this lemma then guarantees that the reduced-order networked system $\hat{\mathbf{\Sigma}}_{\bar{k}}$ satisfies Assumption 2, such that the statements on $\hat{\mathcal{G}}_{\bar{k}}$ and $\hat{\mathcal{G}}_{u,\bar{k}}$ follow from Lemma 8. Moreover, as reduction is performed on the level of the interconnection topology and the subsystem dynamics remain unchanged, Assumption 1 also holds for reduced-order networked systems of arbitrary order. Then, the result (55) on synchronization directly follows from Theorem 5.

To prove the error bound (56), let $\hat{y}_l$ denote the output of a reduced-order system of the form (54) with l subsystems, obtained by a one-step reduction from a networked system of order $l + 1$ and output $\hat{y}_{l+1}$. By combining the results of Theorem 10 and Theorem 11, the error bound for this one-step reduction can be found as

$$\begin{aligned}\|\hat{y}_l - \hat{y}_{l+1}\|_2 &= \|\hat{y}_{a,l} + \hat{y}_{e,l} - (\hat{y}_{a,l+1} + \hat{y}_{e,l+1})\|_2 \\ &= \|\hat{y}_{e,l} - \hat{y}_{e,l+1}\|_2 \\ &\leq 2(M)_{ll}\sqrt{\pi_l^c \pi_l^o}\|u\|_2.\end{aligned} \tag{57}$$

Here, the final equality follows from $\hat{y}_{a,l} = \hat{y}_{a,l+1}$ in Theorem 10, whereas the inequality follows from the one-step error bound on the edge system in Theorem 11. Then, by defining $\hat{y}_{\bar{n}} = y$, the application of the triangle inequality leads to

$$\|\hat{y}_{\bar{k}} - y\|_2 = \left\|\sum_{l=\bar{k}}^{\bar{n}-1}(\hat{y}_l - \hat{y}_{l+1})\right\|_2 \leq \sum_{l=\bar{k}}^{\bar{n}-1}\|\hat{y}_l - \hat{y}_{l+1}\|_2 \tag{58}$$

yielding, after the substitution of (57), the error bound (56) for the reduced-order networked system $\hat{\mathbf{\Sigma}}_{\bar{k}}$ in (54). ■

As shown by Theorem 12, the reduced-order networked system obtained by clustering of subsystems is guaranteed to synchronize. Thus, stability properties of the original networked system have been preserved. Moreover, an *a priori* error bound is available, providing a direct measure of the quality of the reduced-order networked system.

Remark 8: It is stressed that the result on the preservation of synchronization in Theorem 12 does not require the assumption of bidirectionality (in fact, this property can be proven for any graph satisfying only point 2. in Assumption 2). Moreover, the reduction procedure discussed in Section V can be applied as long as solutions $\tilde{\Pi}^c$ and $\tilde{\Pi}^o$ to (21) and (23) exist (for which bidirectionality provides a sufficient, but not necessary, condition). Even though the error bound (56) is not guaranteed for systems that do not satisfy the assumption on bidirectionality, a good approximation can be expected due to the continuous dependence of the solutions of $\mathbf{\Sigma}$ in (6) on the interconnection weights w_{ij}. Here, an arc with zero weight can be approximated by an arc with arbitrarily small positive weight, at least on a finite time interval. ◁

VI. Illustrative Example

To illustrate the clustering-based model reduction procedure developed in Section V, the thermal model of several adjacent rooms in a building is considered. Following [21], each room is modeled as a two thermal-mass system given as

$$\begin{aligned} C_1\dot{T}_1^i &= R_{\text{int}}^{-1}\left(T_2^i - T_1^i\right) + R_{\text{out}}^{-1}\left(T_{\text{env}} - T_1^i\right) + P_i \\ C_2\dot{T}_2^i &= R_{\text{int}}^{-1}\left(T_1^i - T_2^i\right)\end{aligned} \tag{59}$$

where T_1^i and T_2^i are the temperature of the fast thermal mass C_1 and slow thermal mass C_2, respectively (i.e., $C_2 > C_1$). Here, the slow thermal mass represents solid elements such as walls and furniture, whereas the fast thermal mass mainly models the air inside the room. In (59), R_{int} is the thermal resistance between these two thermal masses and R_{out} is the thermal resistance of the outer walls. Moreover, P_i represents the power supplied to the room through external inputs such as heaters and the heat exchange with neighbouring rooms and T_{env} is the environmental temperature. Then, after choosing the state $x_i^T = [T_1^i \; T_2^i]$, the input $v_i = P_i - R_{\text{out}}^{-1}T_{\text{env}}$ and output $z_i = T_1^i$, it can be checked that (59) can be written in the form (8), where

$$\begin{aligned} Q &= \begin{bmatrix} C_1 & 0 \\ 0 & C_2 \end{bmatrix}, \quad J = 0 \\ R &= \frac{1}{R_{\text{int}}C_1C_2}\begin{bmatrix} \frac{C_2}{C_1} & 1 \\ 1 & \frac{C_1}{C_2} \end{bmatrix} - \frac{1}{R_{\text{out}}C_1^2}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\end{aligned} \tag{60}$$

and $B = [C_1^{-1} \; 0]^T$. As a result, Assumption 1 is satisfied. A corridor of six rooms is considered as in Fig. 2, such that the interconnection between the rooms can be written in the form (2) with the weights given by $w_{i,i+1} = w_{i+1,i} = R_{\text{wall}}^{-1}$ for $i \in \{1, 2, 4, 5\}$, $w_{34} = w_{43} = (\kappa R_{\text{wall}})^{-1}$ and $w_{ij} = w_{ji} = 0$ otherwise. Here, R_{wall} represents the nominal thermal resistances of the walls dividing the rooms, where it is remarked that the resistance of the third wall (i.e., the wall between rooms 3 and 4) can be adapted through the parameter $\kappa > 0$.

The control of the temperature in the third room is of interest, and it is assumed that this room has a heater that supplies the power P_h. Then, the external inputs to the network of rooms is given by $u = [P_h \; T_{\text{env}}]^T$ and the corresponding input matrix

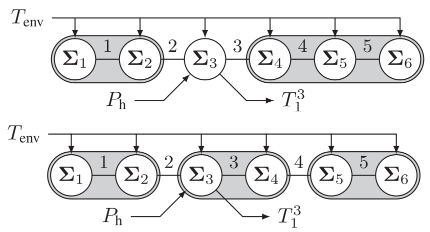

Fig. 2. Path graph representing a corridor of six rooms and the clusters obtained by a three-step reduction for $\kappa = 1$ (left) and $\kappa = 0.25$ (right).

TABLE I
NUMERICAL VALUES OF THE CRITERION $\bar{\pi}_i := (M)_{ll}\sqrt{\pi_l^c \pi_l^o}$ FOR $\kappa = 1$

l	1	2	3	4	5
$10^2\bar{\pi}_i$	0.145	0.951	1.267	0.337	0.092

in the interconnection (2) yields $G = [e_3 \;\; R_{\rm out}^{-1}\mathbf{1}_6]$. Finally, a temperature measurement of the third room is available, such that $y = T_1^3$ and $H = e_3^T$. By combining this interconnection structure with the room dynamics, the networked system has the form (6). Here, it is remarked that the interconnection structure satisfies Assumption 2, where it is recalled this assumption allows for the interconnection topology to have nonequal weights (i.e., the thermal resistances of the walls between different rooms may vary). However, the internal dynamics of each room (that satisfies Assumption 1) is required to be equal. The parameters are taken as $C_1 = 4.35 \cdot 10^4$ J/K, $C_2 = 9.24 \cdot 10^6$ J/K, $R_{\rm int} = 2.0 \cdot 10^{-3}$ K/W, $R_{\rm out} = 23 \cdot 10^{-3}$ K/W and $R_{\rm wall} = 16 \cdot 10^{-3}$ K/W.

After taking $\kappa = 1$, the generalized edge controllability Gramian and generalized edge observability Gramian are computed through (21) and (23), respectively, hereby minimizing their trace. Then, by forming the products $(M)_{ll}^2 \pi_l^c \pi_l^o$, hereby using (41), (42) and recalling that $M = L_e^{-1}$, it follows that edge 5 has the smallest influence on the input-output behavior of the networked system, followed by edges 1 and 4 (see Table I). Consequently, a three-step reduction leads to the clustering as in the left graph of Fig. 2, where it is noted that the rightmost cluster is formed in two steps. Thus, the two leftmost rooms as well as the three rightmost rooms are approximated as a single room each. When the thermal resistance of the wall between rooms 3 and 4 is decreased by choosing $\kappa = 0.25$, a different clustering is obtained, see the right graph in Fig. 2. Even though this corresponds to the intuition that rooms that are strongly coupled (i.e., the wall between them has only a small thermal resistance) show similar behavior, it is stressed that the location of external inputs and outputs also plays an important role. As an example for the case $\kappa = 0.25$, the algorithm yields the large cluster consisting of rooms three to six if the output is changed to a measurement of the temperature in the first room rather than room three.

By Theorem 12, synchronization of the reduced-order networked system (for $u = 0$) is guaranteed. Moreover, Fig. 3 provides a comparison of the frequency response functions (for input P_h, output T_1^3, and $\kappa = 1$) of the original networked system $\mathbf{\Sigma}$ and the reduced-order networked system $\hat{\mathbf{\Sigma}}_3$, showing a close approximation. In fact, it can be shown that the response

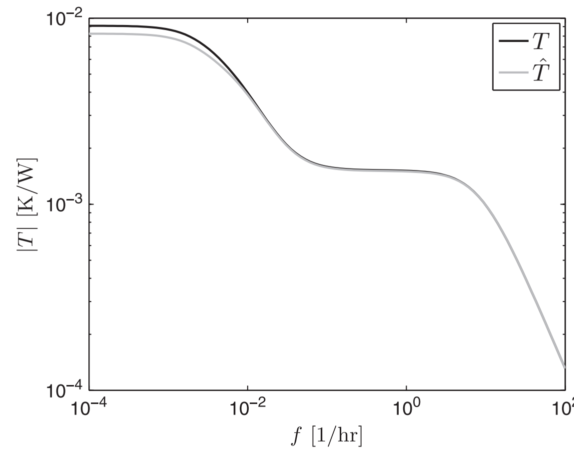

Fig. 3. Comparison of the magnitude of the frequency response functions T of $\mathbf{\Sigma}$ and $\hat{T}$ of $\hat{\mathbf{\Sigma}}_3$ (from input P_h to output T_1^3) for $\kappa = 1$.

with respect to the input $T_{\rm env}$ is matched perfectly, which is related to the fact that this input acts identically on all subsystems and therefore is related to the average system $\mathbf{\Sigma}_a$ in (19) only, which is preserved during the reduction. Finally, Theorem 12 provides an error bound $2\sum_{l=3}^{5}(M)_{ll}\sqrt{\pi_l^c \pi_l^o} = 11.4 \cdot 10^{-3}$, whereas the real error is computed (through the $\mathcal{H}_\infty$-norm of the error system) as $0.849 \cdot 10^{-3}$. The conservatism in the error bound is mainly due to the fact that a diagonal structure is enforced in the generalized edge Gramians in Definitions 4 and 5.

VII. CONCLUSION

In this paper, a model reduction procedure for networked systems with a tree topology and passive subsystems is developed. Based on a controllability and observability analysis of corresponding *edge* systems, clusters of subsystems are identified and subsequently aggregated. As a result, the reduced-order network allows for a convenient physical interpretation. Moreover, it is shown that the property of synchronization is preserved during reduction and a computable *a priori* bound on the reduction error is available.

A natural focus for future research is in the extension of these results towards interconnection topologies including cycles and nonidentical subsystems. Such extensions are however not straightforward and the following obstacles are faced.

It is clear that the concept of clustering and the corresponding projection-based approach discussed in this paper are not limited to graphs with a tree structure. However, the definition of the (generalized) edge Gramians, which are crucial in determining which subsystems to cluster, is an open problem for systems with cycles. Namely, for such systems, the edge Laplacian L_e will have a number of zero eigenvalues that are related to the existence of redundant coordinates in the corresponding edge system and the interdependence between these coordinates (due to the presence of cycles) will affect the definition of the edge Gramians. More specifically, regular (i.e., non-generalized) Gramians are only uniquely defined for asymptotically stable systems and, since the generalized edge Gramians provide bounds on regular Gramians, the current definition might not be suitable for systems with cycles. As an additional complication, it is recalled that any definition of generalized edge Gramians for systems with cycles should feature a property similar to that in Theorem 9 in order to enable model reduction.

In the extension towards networked systems with non-identical subsystems, a major challenge lies in the computation of relevant controllability and observability measures. Namely, in such a case, a decomposition of such measures in terms related to the graph structure and subsystems, respectively, as in Theorem 7, does no longer hold. In addition, the representation of a cluster of non-identical subsystems needs to be defined.

APPENDIX

Before proving the error bound in Theorem 11, the following technical lemma is stated, which relates properties of the interconnection topology to the gain of the networked system.

Lemma 13: Consider an asymptotically stable system

$$\begin{aligned}\dot{x} &= (I \otimes A - \tilde{L} \otimes BC)x + (\tilde{G} \otimes B)u \\ y &= (\tilde{H} \otimes C)x\end{aligned} \tag{61}$$

in which the subsystems satisfy Assumption 1. Moreover, consider the accompanying system

$$\dot{\tilde{x}} = -\tilde{L}\tilde{x} + \tilde{G}\tilde{u}, \quad \tilde{y} = \tilde{H}\tilde{x} \tag{62}$$

and assume that $-\tilde{L}$ is Hurwitz. If there exists a function $\tilde{V}(\tilde{x}) = \tilde{x}^T\tilde{K}\tilde{x}$ with $\tilde{K} = \tilde{K}^T \succcurlyeq 0$ and $\gamma > 0$ such that

$$\dot{\tilde{V}}(\tilde{x}) \leq \gamma^2|\tilde{u}|^2 - |\tilde{y}|^2 \tag{63}$$

holds along all trajectories of (62), then there exists a function $V(x) = x^T K x$ such that

$$\dot{V}(x) \leq \gamma^2|u|^2 - |y|^2 \tag{64}$$

holds along all trajectories of (61). In particular, K can be chosen as $K = \tilde{K} \otimes Q$, with Q the energy function of the subsystems as in (8).

Proof: In order to prove this lemma, it is noted that (63) can be written as

$$\begin{bmatrix}\tilde{x}\\ \tilde{u}\end{bmatrix}^T \begin{bmatrix}-\tilde{L}^T\tilde{K} - \tilde{K}\tilde{L} + \tilde{H}^T\tilde{H} & \tilde{K}\tilde{G}\\ \tilde{G}^T K & -\gamma^2 I\end{bmatrix}\begin{bmatrix}\tilde{x}\\ \tilde{u}\end{bmatrix} \leq 0. \tag{65}$$

Consequently, (63) holds if and only if the matrix given in (65) is negative semi-definite (see also the theory on dissipative systems and the bounded real lemma in, e.g., [8]). By the Schur complement, this is equivalent to

$$-\tilde{L}^T\tilde{K} - \tilde{K}\tilde{L} + \tilde{H}^T\tilde{H} - \gamma^{-2}\tilde{K}\tilde{G}\tilde{G}^T\tilde{K} \preccurlyeq 0. \tag{66}$$

Next, the matrix

$$\begin{aligned}\Lambda := {} & (I \otimes A - \tilde{L} \otimes BC)^T(\tilde{K} \otimes Q)\\ & + (\tilde{K} \otimes Q)(I \otimes A - \tilde{L} \otimes BC)\\ & + (\tilde{H} \otimes C)^T(\tilde{H} \otimes C)\\ & - \gamma^{-2}(\tilde{K} \otimes Q)(\tilde{G} \otimes B)(\tilde{G} \otimes B)^T(\tilde{K} \otimes Q)\end{aligned} \tag{67}$$

is considered, which has the same structure as the left-hand-side of (66) [but for the system (61)]. Thus, by the equivalence discussed above, (64) holds if Λ satisfies $\Lambda \preccurlyeq 0$. Then, expansion of the Kronecker products and substitution of $A = (J - R)Q$ and $C = B^T Q$, hereby noting that $J = -J^T$, leads to

$$\begin{aligned}\Lambda = {} & -2(\tilde{K} \otimes QRQ)\\ & + (-\tilde{L}^T\tilde{K} - \tilde{K}\tilde{L} + \tilde{H}^T\tilde{H} - \gamma^{-2}\tilde{K}\tilde{G}\tilde{G}^T\tilde{K}) \otimes C^T C\end{aligned} \tag{68}$$

such that the substitution of (66) in (68) gives $\Lambda \preccurlyeq -2(\tilde{K} \otimes QRQ) \preccurlyeq 0$. Here, the latter inequality follows from the properties $\tilde{K} \succcurlyeq 0$ and $R \succcurlyeq 0$, proving the lemma. ■

Now, the proof of Theorem 11 can be stated as follows.

Proof of Theorem 11: To prove the theorem, the error between the edge system $\mathbf{\Sigma}_e$ in (20) and the reduced-order edge system $\hat{\mathbf{\Sigma}}_e$ in (40) has to be considered. However, motivated by the results of Lemma 13, the accompanying systems defined by their respective interconnection topologies will be exploited. In particular, instead of the edge system (20), the dynamics

$$\dot{\tilde{x}}_e = -L_e\tilde{x}_e + G_e\tilde{u}, \quad \tilde{y}_e = H_e\tilde{x}_e \tag{69}$$

will be used, where $\tilde{x}_e \in \mathbb{R}^{\bar{n}-1}$. However, it will turn out that it is more convenient to write (69) in a different form, which is obtained by pre-multiplying the dynamics of (69) with $M = L_e^{-1}$ as

$$M\dot{\tilde{x}}_e = -I\tilde{x}_e + G_f\tilde{u}, \quad \tilde{y}_e = H_e\tilde{x}_e. \tag{70}$$

Here, $G_f = L_e^{-1}G_e = (E^T F)^{-1}G_e$ is the same external input matrix as that of the dual edge system (22). Before defining the accompanying system for the reduced-order edge system $\hat{\mathbf{\Sigma}}_e$ in (40), it is recalled that the interconnection topology of $\hat{\mathbf{\Sigma}}_e$ is characterized by $\hat{L}_e$. Then, by considering the partitioning of $M = L_e^{-1}$ in (49), the relations for the inverse of a partitioned matrix (see, e.g., [17]) yield

$$\begin{bmatrix}M_{11} & M_{12}\\ M_{21} & M_{22}\end{bmatrix} = \begin{bmatrix}M_{11} & -M_{11}L_{e,12}L_{e,22}^{-1}\\ -M_{22}^{-1}L_{e,21}L_{e,11}^{-1} & M_{22}\end{bmatrix}. \tag{71}$$

Here, $M_{11} = (L_{e,11} - L_{e,12}L_{e,22}L_{e,21})^{-1} = \hat{L}_e^{-1}$, where the latter equality follows from (44). Similarly, after introducing the partitioning

$$G_f = \begin{bmatrix}G_{f,1}\\ G_{f,2}\end{bmatrix}, \quad H_e = \begin{bmatrix}H_{e,1} & H_{e,2}\end{bmatrix} \tag{72}$$

and observing that $G_f = MG_e$ [see (20)], it can be seen by comparison of (71) and (45) that $G_{f,1} = M_{11}\hat{G}_e$. After using $H_e = H_f M$ [see (22)] and (46), it then follows that the accompanying system to $\hat{\mathbf{\Sigma}}_e$ in (32) can be written as

$$M_{11}\dot{\tilde{\xi}}_e = -I\tilde{\xi}_e + G_{f,1}\tilde{u}, \quad \tilde{\hat{y}}_e = H_{e,1}\tilde{\xi}_e. \tag{73}$$

In the remainder of this proof, the error between the systems (70) and (73) will be considered, hereby exploiting ideas from a frequency-domain approach in [12]. Then, in order to write the transfer functions T of (70) and $\hat{T}$ of (73) in a convenient form, the functions φ and Δ are introduced as

$$\varphi(s) = (sM_{11} + I)^{-1} \tag{74}$$

$$\Delta(s) = sM_{22} + I - s^2 M_{21}\varphi(s)M_{12}. \tag{75}$$

Using these definitions, the expressions for the inverse of a partitioned matrix lead to

$$(sM + I)^{-1} = \begin{bmatrix} \varphi(s) & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} -s\varphi(s)M_{12} \\ I \end{bmatrix} \Delta^{-1}(s) \begin{bmatrix} -M_{21}s\varphi(s) & I \end{bmatrix} \tag{76}$$

from which it follows that $T(s) - \hat{T}(s) = \tilde{H}(s)\Delta^{-1}(s)\tilde{G}(s)$, where:

$$\tilde{H}(s) = -H_{e,1}s\varphi(s)M_{12} + H_{e,2} \tag{77}$$

$$\tilde{G}(s) = -M_{21}s\varphi(s)G_{f,1} + G_{f,2}. \tag{78}$$

This thus expresses the error (in Laplace domain) between the accompanying systems of the edge system and reduced-order edge system. The size of this error is quantified by using the $\mathcal{H}_\infty$ norm, which is given as

$$\left\| T(s) - \hat{T}(s) \right\|_{\mathcal{H}_\infty} = \sup_{\omega \in \mathbb{R}} \sigma_{\max}\left(T(j\omega) - \hat{T}(j\omega) \right) \tag{79}$$

and were $\sigma_{\max}(\cdot)$ represents the largest singular value. By noting that the transfer function Δ in (75) and the product $\tilde{G}\tilde{G}^H$ are scalar [see (78)], it follows that (the squared value of) the maximum singular value in (79) can be written as

$$\sigma^2_{\max}\left(T(j\omega) - \hat{T}(j\omega) \right) = \frac{\tilde{G}(j\omega)\tilde{G}^H(j\omega)}{\Delta(j\omega)\Delta^H(j\omega)} \lambda_{\max}\left(\tilde{H}(j\omega)\tilde{H}^H(j\omega) \right). \tag{80}$$

Here, T^H denotes the Hermitian transpose satisfying $T^H(j\omega) = (T(-j\omega))^T$ and $\lambda_{\max}(\cdot)$ represents the largest eigenvalue. However, it is clear from (77) that the matrix $\tilde{H}\tilde{H}^H$ is of rank one, such that its only nonzero eigenvalue (hence, $\lambda_{\max}$) reads $\tilde{H}^H\tilde{H}$, leading to

$$\sigma^2_{\max}\left(T(j\omega) - \hat{T}(j\omega) \right) = \frac{\tilde{G}(j\omega)\tilde{G}^H(j\omega)\tilde{H}^H(j\omega)\tilde{H}(j\omega)}{\Delta(j\omega)\Delta^H(j\omega)}. \tag{81}$$

It is remarked that (81) is a real-valued function.

In order to find a bound on the magnitude of the frequency response function in (81), several terms are considered separately. First, it follows from the definition of $\tilde{G}$ in (78) that

$$\tilde{G}(j\omega)\tilde{G}^H(j\omega) = \begin{bmatrix} -M_{21}j\omega\varphi(j\omega) & I \end{bmatrix} \begin{bmatrix} G_{f,1} \\ G_{f,2} \end{bmatrix} \times \begin{bmatrix} G_{f,1} \\ G_{f,2} \end{bmatrix}^T \begin{bmatrix} j\omega\varphi^H(-j\omega)M_{21}^T \\ I \end{bmatrix}. \tag{82}$$

At this point, it is recalled that the generalized edge controllability Gramian $\tilde{\Pi}^c$ in Definition 4 satisfies inequality (21). Pre- and post-multiplication of (21) with $M = L_e^{-1}$ and M^T, respectively, leads to

$$\tilde{\Pi}^c M^T + M\tilde{\Pi}^c - MG_eG_e^TM^T = \tilde{\Pi}^c M^T + M\tilde{\Pi}^c - G_fG_f^T \succcurlyeq 0. \tag{83}$$

Then, by exploiting the partitioning of $\tilde{\Pi}^c$ in (41), M in (49), and G_f in (72), the use of (83) in (82) leads to

$$\tilde{G}(j\omega)\tilde{G}^H(j\omega) \leq \nabla(j\omega)\tilde{\Pi}^c_2 + \tilde{\Pi}^c_2\nabla^H(j\omega). \tag{84}$$

where the function ∇ is defined as

$$\nabla(j\omega) = M_{22} - M_{21}j\omega\varphi(j\omega)M_{12}. \tag{85}$$

A similar procedure can be performed for the observability case, hereby using the definition of the generalized edge observability Gramian $\tilde{\Pi}^o$ in (23). Then, again exploiting the definition (85), it can be shown that

$$\tilde{H}^H(j\omega)\tilde{H}(j\omega) \leq \nabla^H(j\omega)\tilde{\Pi}^o_2 + \tilde{\Pi}^o_2\nabla(j\omega). \tag{86}$$

Now, by collecting the bounds on $\tilde{G}(j\omega)\tilde{G}^H(j\omega)$ in (84) and $\tilde{H}^H(j\omega)\tilde{H}(j\omega)$ in (86) (and recalling that all frequency response functions are scalar), it follows that the magnitude of the error in (81) can be written as

$$\sigma^2_{\max}\left(T(j\omega) - \hat{T}(j\omega) \right) \leq \tilde{\Pi}^c_2\tilde{\Pi}^o_2\delta(\omega) \tag{87}$$

where the real-valued function δ is defined as

$$\delta(\omega) = \frac{\left(\nabla(j\omega) + \nabla^H(j\omega)\right)^2}{\Delta(j\omega)\Delta^H(j\omega)}. \tag{88}$$

In the remainder of the proof, it will be shown that δ satisfies

$$\sup_{\omega \in \mathbb{R}} \delta(\omega) = \delta(0) = (2M_{22})^2 \tag{89}$$

where M_{22} is the discarded block of the matrix M in (49).

In order to show that M_{22} is positive, it is remarked that the assumption that $w_{ij} > 0$ if and only if $w_{ji} > 0$ ensures that the conditions of Lemma 4 hold and that the matrix F as in (9) can be written as (16). Then, it follows that:

$$M = L_e^{-1} = \left(E^TF\right)^{-1} = D_2\left(E^TD_1^{-1}E\right)^{-1} \tag{90}$$

where $D_1 \succ 0$. Consequently $\left(E^TD_1^{-1}E\right)^{-1}$ is a positive definite matrix and its diagonal elements are positive. Since $D_2 \succ 0$ (by Lemma 4), it follows that the diagonal elements of M in (90) are positive and $M_{22} > 0$ [see (49)].

Next, several intermediate steps will be taken to prove the equality (89). First, Lemma 4 is exploited again and the matrices D_1 and D_2 in (16) are partitioned according to the partitioning of the matrices E and F in (33) as

$$D_1 = \mathrm{diag}\{D_{1,0}, D_{1,i}, D_{1,j}\},\ D_2 = \mathrm{diag}\{D_{2,0}, D_{2,l}\} \tag{91}$$

such that the equality (16) and the partitioning (33) directly leads to $E_{00} = D_{1,0}F_{00}D_{2,0}$, $E_{i0} = D_{1,i}F_{i0}D_{2,0}$ and $E_{j0} = D_{1,j}F_{j0}D_{2,0}$. The substitution of this in the expression for the incidence matrix for the reduced-order graph $\hat{E}$ in (34) in Lemma 8 leads to

$$\hat{E} = \begin{bmatrix} E_{00} \\ E_{i0} + E_{j0} \end{bmatrix} = \begin{bmatrix} D_{1,0} & 0 \\ 0 & D_{1,i}\frac{w_{ij}+w_{ji}}{w_{ji}} \end{bmatrix} \begin{bmatrix} F_{00} \\ \frac{w_{ji}}{w_{ij}+w_{ji}}F_{i0} + \frac{w_{ij}}{w_{ij}+w_{ji}}F_{j0} \end{bmatrix} D_{2,0} \tag{92}$$

where the property $D_{1,i}w_{ij} = D_{1,j}w_{ji}$ is used, which can be concluded from the construction of the matrix D_1 as in (17). It is noted that the second matrix in the multiplication at the right-hand side of (92) equals $\hat{F}$ [see (35)], such that (92) provides a relation of the form

$$\hat{E} = \hat{D}_1 \hat{F} \hat{D}_2 \tag{93}$$

where $\hat{D}_2$ can be chosen as $D_{2,0}$. This relation will be exploited later.

As a second intermediate step to show (89), the relation $E = D_1 F D_2$ in (16) is again exploited. Namely, it follows from (16) that the matrix $D_2^{-1/2} L_e D_2^{1/2} = D_2^{-1/2} E^T D_1^{-1} E D_2^{-1/2}$ is symmetric. By using this fact, as well as the partitioning of L_e in (43) and the partitioning of D_2 in (91), it is readily shown that

$$D_{2,l}^{-\frac{1}{2}} L_{e,21} D_{2,0}^{\frac{1}{2}} = \left(D_{2,0}^{-\frac{1}{2}} L_{e,12} D_{2,l}^{\frac{1}{2}} \right)^T. \tag{94}$$

The third and final intermediate step is based on the relation (93), where it is recalled that $\hat{D}_2$ can be chosen as $D_{2,0}$ as in (91). Then, the factorization $\hat{L}_e = \hat{E}^T \hat{F}$ and the relation (93) imply that $D_{2,0}^{-1/2} \hat{L}_e D_{2,0}^{1/2}$ is a symmetric matrix (see above for the high-order case), such that it admits an eigenvalue decomposition $U \Lambda U^T$. For the reduced-order edge Laplacian, this leads to the eigenvalue decomposition

$$\hat{L}_e = D_{2,0}^{\frac{1}{2}} U \Lambda U^T D_{2,0}^{-\frac{1}{2}} \tag{95}$$

where it is noted that all eigenvalues (on the diagonal of Λ) are real-valued and positive (by Lemma 3) and U is an orthogonal matrix.

Now, returning to the proof of the bound (89), the function ∇ in (85) is considered. By using the definition of φ in (74), it can be shown that ∇ can be written as

$$\nabla(j\omega) = \left(M_{22} - M_{21} M_{11}^{-1} M_{12} \right) + M_{21} M_{11}^{-1} (j\omega I + \hat{L}_e)^{-1} M_{11}^{-1} M_{12}. \tag{96}$$

The use of the eigenvalue decomposition (95) in (96) yields

$$\nabla(j\omega) = \left(M_{22} - M_{21} M_{11}^{-1} M_{12} \right) + \sum_{i=1}^{\bar{n}-2} \frac{c_i}{j\omega + \lambda_i} \tag{97}$$

where the constants c_i are given as

$$c_i = M_{21} M_{11}^{-1} D_{2,0}^{\frac{1}{2}} U_i U_i^T D_{2,0}^{-\frac{1}{2}} M_{11}^{-1} M_{12} \tag{98}$$

and with U_i the i-th column of U. By exploiting the expressions for the inverse of a partitioned matrix, it follows from (49) that $M_{11}^{-1} M_{12} = -L_{e,12} L_{e,22}^{-1}$ and $M_{21} M_{11}^{-1} = -L_{e,22}^{-1} L_{e,21}$ [see also (71)], such that the expression for c_i in (98) can be written as

$$c_i = L_{e,22}^{-1} L_{e,21} D_{2,0}^{\frac{1}{2}} U_i U_i^T D_{2,0}^{-\frac{1}{2}} L_{e,12} L_{e,22}^{-1}. \tag{99}$$

At this point, it is recalled that $L_{e,22}$ is scalar due to the one-step reduction. Then, by noting that $D_{2,l}$ in the partitioned diagonal matrix D_2 in (91) is scalar (and strictly positive by Lemma 4), it follows that:

$$c_i = L_{e,22}^{-1} D_{2,l}^{-\frac{1}{2}} L_{e,21} D_{2,0}^{\frac{1}{2}} U_i U_i^T D_{2,0}^{-\frac{1}{2}} L_{e,12} D_{2,l}^{\frac{1}{2}} L_{e,22}^{-1} \tag{100}$$

$$= \left| U_i^T D_{2,0}^{-\frac{1}{2}} L_{e,12} D_{2,l}^{\frac{1}{2}} L_{e,22}^{-1} \right|^2 \tag{101}$$

where the relation (94) is used to obtain the latter equality. Consequently, c_i is nonnegative, i.e., $c_i \geq 0$. This condition will turn out to be crucial in proving the result (89).

To prove (89), the numerator and denominator of δ in (88) are considered separately. First, the (square root of) the numerator can be written as

$$\nabla(j\omega) + \nabla^H(j\omega) = 2\left(M_{22} - M_{21} M_{11}^{-1} M_{12} \right) + \sum_{i=1}^{\bar{n}-2} \frac{2 c_i \lambda_i}{\omega^2 + \lambda_i^2}. \tag{102}$$

By noting that $M_{22} - M_{21} M_{11}^{-1} M_{12} = L_{e,22}^{-1} = 1/L_{e,22}$ is strictly positive and recalling that $c_i \lambda_i \geq 0$, it is clear that (102) is a real-valued positive function which decreases as ω increases. Consequently

$$\sup_{\omega \in \mathbb{R}} \nabla(j\omega) + \nabla^H(j\omega) = \nabla(0) + \nabla^H(0) = 2M_{22}. \tag{103}$$

Next, the denominator of δ in (88) is considered. By comparing the definitions of Δ in (75) and ∇ in (85), it follows that $\Delta(j\omega) = I + j\omega\nabla(j\omega)$. Then, by substitution of (97) and explicitly expressing $\Delta(j\omega)\Delta^H(j\omega)$, it can be shown that $\Delta(j\omega)\Delta^H(j\omega)$ is a real-valued nondecreasing positive function, such that

$$\inf_{\omega \in \mathbb{R}} \Delta(j\omega)\Delta^H(j\omega) = \Delta(0)\Delta^H(0) = I. \tag{104}$$

Herein, the properties $\lambda_i > 0$ and $c_i \geq 0$ are crucial in deriving the result. Then, the use of the bounds (103) and (104) leads to

$$\sup_{\omega \in \mathbb{R}} \delta(\omega) \leq \frac{\sup_{\omega \in \mathbb{R}} \left(\nabla(j\omega) + \nabla^H(j\omega) \right)^2}{\inf_{\omega \in \mathbb{R}} \Delta(j\omega)\Delta^H(j\omega)} = \frac{\left(\nabla(0) + \nabla^H(0) \right)^2}{\Delta(0)\Delta^H(0)} = \delta(0) = (2M_{22})^2 \tag{105}$$

which proves the result (89).

The substitution of (89) in (87) leads to

$$\sigma_{\max}\left(T(j\omega) - \hat{T}(j\omega) \right) \leq 2M_{22} \left(\tilde{\Pi}_2^c \tilde{\Pi}_2^o \right)^{\frac{1}{2}} \tag{106}$$

providing a bound on the gain of the error system formed by the accompanying system of $\mathbf{\Sigma}_e$ in (70) and the accompanying system of $\hat{\mathbf{\Sigma}}_e$ in (73). As this result is independent on the internal realization of the systems, it also holds for the representation (69) of the accompanying system of $\mathbf{\Sigma}_e$. Choosing a similar representation for the accompanying system of $\hat{\mathbf{\Sigma}}_e$, it is clear that the bound (106) holds for an error system of the form (62) with system matrices

$$\tilde{L} = \begin{bmatrix} L_e & 0 \\ 0 & \hat{L}_e \end{bmatrix}, \; \tilde{G} = \begin{bmatrix} L_e G_f \\ \hat{L}_e G_{f,1} \end{bmatrix}$$
$$\tilde{H} = \begin{bmatrix} H_e & -H_{e,1} \end{bmatrix}. \tag{107}$$

At this point, it is recalled that the bounded real lemma (see, e.g., [8]) guarantees the equivalence between a frequency-domain gain bound of the form (106) and the existence of a matrix $\tilde{K}$ such that (63) holds for $\gamma = 2M_{22}(\tilde{\Pi}_2^c \tilde{\Pi}_2^o)^{\frac{1}{2}}$. Then, the error bound (50) follows from the application of Lemma 13. Here, it is noted that the system (61) with matrices (107) characterizes the error system $\mathbf{\Sigma}_e - \hat{\mathbf{\Sigma}}_e$ and that the inequality (64) is equivalent to the system gain using $\mathcal{L}_2$ signal norms as in (50), see, e.g., [8], [32]. ■

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