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Distributed Design of Locally Stabilizing Controllers for Large-Scale Networked Linear Systems

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***Abstract*— In this paper, we consider designing locally stabilizing controllers, each of which stabilizes each disconnected subsystem, in a distributed manner for large-scale networked linear systems. To this end, we design a low-dimensional hierarchical distributed compensator such that the $\mathcal{L}_2$-performance of the closed-loop system improves as long as that of the locally stabilizing controllers improves. We solve a controller reduction problem where the approximation error of the low-dimensional compensator gets better as long as the performance of the locally stabilizing controllers improves, while preserving the hierarchical distributed structure of the original compensator. Finally, we demonstrate the efficiency of the proposed method through a numerical example of a power network.**

I. Introduction

As technology advances, control system architecture become larger and more complex. For example, in a smart grid, it is required to maintain supply-demand balance involving more than one million consumers and several power plants [1], [2]. In many cases, such large-scale systems are spatially distributed and networked. In view of this, it is crucial to establish a framework for designing distributed control systems [3], [4].

Even though many distributed controller design methods have been developed in the literature [5], they do not allow controller design being done in a distributed manner. In view of this, a notion of *distributed design* is introduced in [6], where a performance limitation of controllers that are designed in a distributed manner is discussed. These results were extended to networked control systems in [7]. In [8], a distributed design method in terms of the $\mathcal{L}_1$-induced norm is developed for positive linear systems. However, we cannot straightforwardly generalize this result to a broader class of systems.

In [9], the authors propose a distributed design method for general networked linear systems. In this method, we propose a hierarchical distributed compensator design such that the $\mathcal{L}_2$-induced norm performance of the overall closed-loop system, whose example is shown in Fig. 1, improves as long as that of each local closed-loop system improves,

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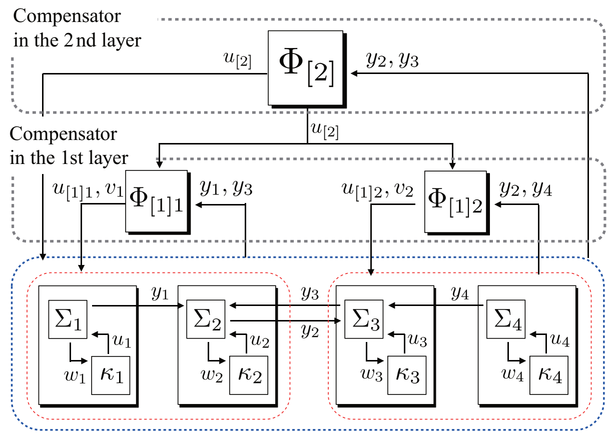


Fig. 1. Example of the hierarchical distributed control system with two layers for a networked system composed of four subsystems $\Sigma_1, \cdots, \Sigma_4$. The family $\{\Phi_{[2]}, \Phi_{[1]1}, \Phi_{[1]2}\}$ is called a hierarchical distributed compensator and κ_i is called a locally stabilizing controller in the sense that it stabilizes the local closed-loop system (Σ_i, κ_i). The notation of signals, e.g., $u_{[2]}$, are defined in Section II.

which enables us distributed design of locally stabilizing controllers. However, the compensator in each layer, e.g., $\Phi_{[1]1}$ and $\Phi_{[1]2}$ in the first layer and $\Phi_{[2]}$ in the second layer in Fig. 1, is necessarily the same dimensional system as the system to be controlled, e.g., the networked system composed of $\Sigma_1, \cdots, \Sigma_4$ in Fig. 1. Thus, the proposed method does not fully comply with practical applications of large-scale networked systems.

Against this background, in this paper, we propose a design method of low-dimensional hierarchical distributed compensators such that the $\mathcal{L}_2$-induced norm performance of the closed-loop system improves as long as that of the local closed-loop system improves. To this end, we take a controller reduction approach with explicit consideration on the influence of the controller approximation error on the closed-loop system. More specifically, supposing that a hierarchical distributed compensator is given by the design method in [9], we find a low-dimensional compensator such that

1) the trajectory of the system state controlled by the low-dimensional compensator gets closer to that controlled by the original compensator as long as the $\mathcal{L}_2$-induced norm performance of local closed-loop system improves, and
2) the low-dimensional compensator has the same hierarchical distributed structure as that of the original

compensator.

The controller reduction problem having the difficulties 1) and 2) cannot be solved by the straightforward use of existing model reduction methods. Regarding to 1), there are no fully developed model reduction methods such that the approximation error improves as long as the performance of a part of the original system improves. Regarding to 2), model reduction methods preserving the network structure of the original system are proposed in the literature, e.g., [10], [11]. However, in [10], it is not necessarily guaranteed that there exist block-diagonal controllability and observability gramians satisfying Lyapunov inequalities in general and the method in [11] is not applicable to general networked systems such as strongly interconnected networked systems. To overcome the difficulties 1) and 2), in this paper, we explicitly utilize the hierarchical distributed structure of the closed-loop system. More specifically, taking into account the inherent hierarchy of information transmission, which can be represented as the block-triangular structure of a coordinate-transformed closed-loop system, we show that the approximation error of compensators in upper layers does not affect those in lower layers. Next, using an orthogonal projection [12], we clarify the relation between the approximation error and the performance degradation of the closed-loop system. Finally, the efficiency of the proposed method is demonstrated through a numerical example of a power network.

The organization of this paper is as follows: In Section II, we first review the distributed design method proposed in [9] and we formulate a controller reduction problem to design a low-dimensional hierarchical distributed compensator. In Section III, we propose a method to solve the controller reduction problem. Furthermore, we show that the approximation error bound improves as long as the $\mathcal{L}_2$-induced norm performance of the local closed-loop system improves. In Section IV, we demonstrate the efficiency of the proposed method through an example of a power network.

Notation: We denote the set of real numbers by $\mathbb{R}$ and the cardinality of a set $\mathcal{I}$ by $|\mathcal{I}|$. In addition, we denote the n-dimensional identity matrix by I_n, where the subscript n is omitted when no confusion arises. Furthermore, for $\mathbb{N} = \{1,\ldots,N\}$, we denote the block-diagonal matrix having matrices $M_1,\ldots,M_N$ on its diagonal blocks by $\mathrm{dg}(M_i)_{i\in\mathbb{N}}$, where the subscript of $i \in \mathbb{N}$ is omitted when no confusion arises. The $\mathcal{L}_2$-norm of a square integrable function $v(t) : \mathbb{R} \mapsto \mathbb{R}^n$ is defined by $\|v(t)\|_{\mathcal{L}_2} := \left(\int_0^\infty v^{\mathsf{T}}(t)v(t)dt\right)^{\frac{1}{2}}$. The $\mathcal{H}_\infty$-norm of a stable proper transfer matrix G is defined by $\|G(s)\|_{\mathcal{H}_\infty} := \sup_{\omega\in\mathbb{R}} \|G(j\omega)\|$ where $\|\cdot\|$ denotes the induced 2-norm.

II. Problem Formulation

A. Review of Hierarchical Distributed Control

1) Motivation of Hierarchical Distributed Control: In this subsection, we review the hierarchical distributed control proposed in [9], which enables distributed design of locally stabilizing controllers explained below. We deal with networked linear systems composed of N subsystems. For each $i \in \mathbb{N} := \{1,\ldots,N\}$, the dynamics of the ith subsystem is described by

$$\Sigma_i : \begin{cases} \dot{x}_i = A_i x_i + B_i u_i + \sum_{j\neq i} J_{i,j} y_j \\ y_i = C_i x_i \\ w_i = S_i x_i \end{cases} \tag{1}$$

where $x_i \in \mathbb{R}^{n_i}$ is a state, $u_i \in \mathbb{R}^{m_i}$ and $w_i \in \mathbb{R}^{q_i}$ are used for connection to a locally stabilizing controller, and $y_i \in \mathbb{R}^{p_i}$ is an output signal used for interconnection among subsystems. The matrices A_i, B_i, $J_{i,j}$, C_i and S_i are real and of compatible dimensions. Note $J_{i,j} = 0$ if the ith and jth subsystems are disconnected. We assume that y_i and w_i are measureable.

Let us consider a locally stabilizing controller given by

$$\kappa_i : \begin{cases} \dot{\xi}_i = \mathbf{K}_i \xi_i + \mathbf{H}_i w_i \\ u_i = \mathbf{M}_i \xi_i \end{cases} \tag{2}$$

where $\xi_i \in \mathbb{R}^{r_i}$. For simplicity, we take $\xi_i(0) = 0$. Throughout this paper, the symbols in bold font, e.g., $\mathbf{K}_i$ are parameters to be designed. Furthermore, we denote a family of locally stabilizing controllers by $\{\kappa_i\}_{i\in\mathbb{N}}$. If no confusion arises, we omit the subscript $i \in \mathbb{N}$.

Let us suppose the situation that we have $\{\kappa_i\}$ stabilizing the whole network system, and suppose that we change only κ_i. Existing distributed controller design methods, e.g., [5], do not fully fit for this situation from a viewpoint of computational costs for controller design because the methods require us to design all of $\{\kappa_i\}$, but not only κ_i. Thus, it is desired that we can design locally stabilizing controllers in a distributed manner, which we call *distributed design* of locally stabilizing controllers. For example if the subsystems are disconnected, i.e., $J_{i,j} = 0$ for all $i, j \in \mathbb{N}$, we can design κ_i individually by taking into account the local closed-loop system (Σ_i, κ_i), but not the overall system. Furthermore, we clearly see that the $\mathcal{L}_2$-induced norm performance of the whole closed-loop system improves as long as each κ_i improves the performance of the local closed-loop system. However, in general cases, i.e., $J_{i,j} \neq 0$, we have a problem that the distributed design no longer guarantees the stability of the whole closed-loop system. To overcome this problem, we consider compensating the networked system such that the compensated networked system enables distributed design of locally stabilizing controllers.

2) Hierarchical Distributed Compensator Design: In what follows, we use the notation

$$\circ := \sum_{i=1}^{N} \circ_i, \quad \forall \circ \in \{n, m, p, q, r\}$$

and

$$J := \begin{bmatrix} J_{1,1} & \cdots & J_{1,N} \\ \vdots & \ddots & \vdots \\ J_{N,1} & \cdots & J_{N,N} \end{bmatrix} \in \mathbb{R}^{n\times p}. \tag{3}$$

In addition, we define

$$A := \mathrm{dg}(A_i) + J\mathrm{dg}(C_i) \in \mathbb{R}^{n\times n}. \tag{4}$$

Let us introduce a hierarchical structure to networked systems. Let $\mathbb{L} := \{1, \dots, L\}$ represent a set of system layers. Furthermore, we introduce a notation to deal with several subsystems collectively in each layer. We first define a family of index sets $\mathbb{N}_{[1]}, \dots, \mathbb{N}_{[L]}$ by $\{\mathbb{N}_{[l]}\}$ such that

$$N \geq |\mathbb{N}_{[1]}| \geq \cdots \geq |\mathbb{N}_{[L]}| = 1, \ \mathbb{N}_{[l]} = \{1, \dots, |\mathbb{N}_{[l]}|\}. \quad (5)$$

In addition, for each $l \in \{0, \dots, L-1\}$, we define a family of cluster sets $\{\mathbb{C}_{[l]i}\}_{i \in \mathbb{N}_{[l+1]}}$ such that

$$\bigcup_{i \in \mathbb{N}_{[l+1]}} \mathbb{C}_{[l]i} = \mathbb{N}_{[l]}, \quad \mathbb{C}_{[l]i} \cap \mathbb{C}_{[l]j} = \emptyset, \quad i \neq j, \quad (6)$$

where $\mathbb{N}_{[0]}$ is regarded as $\mathbb{N}$. Let $A_{[l]i} \in \mathbb{R}^{n_{[l]i} \times n_{[l]i}}$ denote the principal submatrices of A compatible with subsystems belonging to $\mathbb{C}_{[l-1]i}$, and $J_{[l]i} \in \mathbb{R}^{n_{[l]i} \times p_{[l]i}}$ denote the submatrix of J compatible with interconnection among clusters in $\mathbb{C}_{[l-1]i}$. By definition, it follows that $\sum_{i \in \mathbb{N}_{[l]}} n_{[l]i} = n$ and $\sum_{i \in \mathbb{N}_{[l]}} p_{[l]i} = p$ for each $l \in \mathbb{L}$, and $A_{[L]1} = A$. In the rest of this paper, we regard $A_{[0]i}$ and $n_{[0]i}$ as A_i and n_i for all $i \in \mathbb{N}$.

We give the overall dynamics of the networked system by

$$\Sigma : \begin{cases} \dot{x} = Ax + \mathrm{dg}(B_i)u + \sum_{l=1}^{L} \mathrm{dg}(B_{[l]i})u_{[l]} \\ y = \mathrm{dg}(C_i)x \\ w = \mathrm{dg}(S_i)x + v \end{cases} \quad (7)$$

with $x(0) = x_0$ where $u_{[l]} := [u_{[l]1}^{\mathsf{T}}, \dots, u_{[l]|\mathbb{N}_{[l]}|}^{\mathsf{T}}]^{\mathsf{T}}$, $u_{[l]i} \in \mathbb{R}^{m_{[l]i}}$ and $v := [v_1^{\mathsf{T}}, \dots, v_N^{\mathsf{T}}]^{\mathsf{T}} \in \mathbb{R}^q$ express additional compensation signals from a hierarchical distributed compensator to be explained below, $u := [u_1^{\mathsf{T}}, \dots, u_N^{\mathsf{T}}]^{\mathsf{T}} \in \mathbb{R}^m$ and $w := [w_1^{\mathsf{T}}, \dots, w_N^{\mathsf{T}}]^{\mathsf{T}} \in \mathbb{R}^q$ are used for the interconnection to locally stabilizing controllers, and $y := [y_1^{\mathsf{T}}, \dots, y_N^{\mathsf{T}}]^{\mathsf{T}} \in \mathbb{R}^p$ is used for not only the interconnection among the subsystems, but also the interconnection to the hierarchical distributed compensator. In what follows, the pair $(A_{[l]i}, B_{[l]i})$, which is defined as being compatible with the hierarchical structure of the networked system, is assumed to be stabilizable for any $i \in \mathbb{N}_{[l]}$ and $l \in \mathbb{L}$.

To construct appropriate input signals of $\{u_{[l]}\}_{l \in \mathbb{L}}$ and v in (7) towards distributed design of $\{\kappa_i\}$ in (2), we consider designing a hierarchical distributed compensator given by

$$\Phi_{[l]} : \begin{cases} \dot{\phi}_{[l]} = \mathrm{dg}(\mathbf{E}_{[l]i})\phi_{[l]} + \mathbf{\Gamma}_{[l]}y + \sum_{k=l}^{L} \mathrm{dg}(\mathbf{\Lambda}_{[k]i})u_{[k]} \\ u_{[l]} = \mathrm{dg}(\mathbf{F}_{[l]i})\phi_{[l]} + \mathbf{G}_{[l+1]}\phi_{[l+1]} \end{cases} \quad (8)$$

with $\phi_{[l]}(0) = 0$ and

$$v = \mathrm{dg}(\mathbf{U}_i)\phi_{[1]}$$

where $\mathbf{G}_{[L+1]}$ and $\phi_{[L+1]}$ are regarded as zero, and

$$\begin{array}{lll} \mathbf{E}_{[l]i} \in \mathbb{R}^{n_{[l]i} \times n_{[l]i}}, & \mathbf{\Gamma}_{[l]} \in \mathbb{R}^{n \times p}, & \mathbf{\Lambda}_{[l]i} \in \mathbb{R}^{n_{[l]i} \times m_{[l]i}} \\ \mathbf{F}_{[l]i} \in \mathbb{R}^{m_{[l]i} \times n_{[l]i}}, & \mathbf{G}_{[l+1]} \in \mathbb{R}^{m_{[l]} \times n}, & \mathbf{U}_i \in \mathbb{R}^{q_i \times n_{[l]i}} \end{array}$$

are design parameters. In what follows, we denote the hierarchical distributed compensator by $\{\Phi_{[l]}\}$, the compensated networked system by $(\Sigma, \{\Phi_{[l]}\})$ and the closed-loop system, which is the interconnected system of $(\Sigma, \{\Phi_{[l]}\})$ and $\{\kappa_i\}$ in (2), by $((\Sigma, \{\Phi_{[l]}\}), \{\kappa_i\})$.

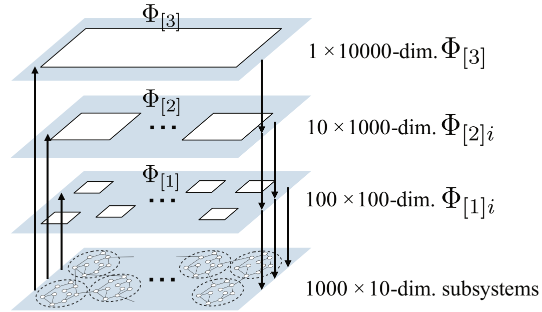


Fig. 2. Example of the hierarchical distributed compensator $\{\Phi_{[l]}\}$ for a networked system composed of 1000 subsystems, each of which is a 10-dimensional system

The hierarchical distributed compensator, which enables us distributed design of locally stabilizing controllers, is provided by the following lemma in [9]:

Lemma 1: Given $\{\mathbb{N}_{[l]}\}$ and $\{\mathbb{C}_{[l]i}\}_{i \in \mathbb{N}_{[l+1]}}$ such that (5) and (6), consider Σ in (7). Consider $\{\kappa_i\}$ in (2). Give $\{\Phi_{[l]}\}$ in (8) with

$$\begin{array}{ll} \mathbf{E}_{[l]i} = \mathrm{dg}(A_{[l-1]j})_{j \in \mathbb{C}_{[l-1]i}}, & \mathbf{\Gamma}_{[l]} = \sum_{k=l}^{L} \mathrm{dg}(J_{[k]i})_{i \in \mathbb{N}_{[k]}} \\ \mathbf{\Lambda}_{[l]i} = B_{[l]i}, \quad \mathbf{G}_{[l+1]} = -\mathrm{dg}(\mathbf{F}_{[l]i}), & \mathbf{U}_i = -S_i \end{array} \quad (9)$$

and $\mathbf{F}_{[l]i}$ stabilizing $A_{[l]i} + B_{[l]i}\mathbf{F}_{[l]i}$. for each $l \in \mathbb{L}$. Then, $((\Sigma, \{\Phi_{[l]}\}), \{\kappa_i\})$ is stable for any κ_i stabilizing (Σ_i, κ_i). Furthermore, consider

$$\begin{bmatrix} \dot{\chi} \\ \dot{\xi} \end{bmatrix} = \begin{bmatrix} \mathrm{dg}(A_i) & \mathrm{dg}(B_i\mathbf{M}_i) \\ \mathrm{dg}(\mathbf{H}_iS_i) & \mathrm{dg}(\mathbf{K}_i) \end{bmatrix} \begin{bmatrix} \chi \\ \xi \end{bmatrix}, \quad \chi(0) = x_0 \quad (10)$$

where $\xi := [\xi_1^{\mathsf{T}}, \dots, \xi_N]^{\mathsf{T}}$ and define

$$\gamma_{[l]} := \left\| \left(sI - \mathrm{dg}(A_{[l]i} + B_{[l]i}\mathbf{F}_{[l]i})\right)^{-1} \mathrm{dg}(J_{[l]i})\mathrm{dg}(C_i) \right\|_{\mathcal{H}_\infty}. \quad (11)$$

Then, $((\Sigma, \{\Phi_{[l]}\}), \{\kappa_i\})$ satisfies

$$\|x(t)\|_{\mathcal{L}_2} \leq \prod_{l=1}^{L} \left(1 + \gamma_{[l]}\right) \|\chi(t)\|_{\mathcal{L}_2} \quad (12)$$

for all $x(0) = x_0 \in \mathbb{R}^n$.

In Lemma 1, $\{\Phi_{[l]}\}$ by (8) and (9) can be designed independently from locally stabilizing controllers. Furthermore, (12) shows that the compensated system $(\Sigma, \{\Phi_{[l]}\})$ has the property that the upper bound of the $\mathcal{L}_2$-induced norm performance of the closed-loop system improves as long as each locally stabilizing controller improves the performance of the system (10), which are the local closed-loop systems *without* interconnection among subsystems. This property enables us distributed design of locally stabilizing controllers.

In Fig. 1, we show an example of the closed-loop system with the notation of signals. In this figure, $\Phi_{[1]1}$ and $\Phi_{[1]2}$ denote the $n_{[1]1}$- and $n_{[1]2}$-dimensional compositional unit of $\Phi_{[1]}$ in (8). Similarly to this, we denote $\Phi_{[l]i}$ by the $n_{[l]i}$-dimensional unit of $\Phi_{[l]}$.

Note that the sum of $n_{[l]i}$ must coincide with the dimension of the networked system, i.e., $\sum_{i \in \mathbb{N}_{[l]}} n_{[l]i} = n$. In Fig. 2, we show an example of the hierarchical distributed compensator for a networked system composed of 1000 subsystems, each of which is a 10-dimensional system. As we see from this example, the dimension of $\Phi_{[l]i}$ becomes the larger in upper layers. Thus, the hierarchical distributed compensator is not practical for large-scale networked systems due to high computational costs for implementation.

B. Low-dimensional Hierarchical Distributed Control Problem

In this subsection, we consider designing a low-dimensional hierarchical distributed compensator which enables us distributed design of locally stabilizing controllers. More specifically, let us consider

$$\hat{\Phi}_{[l]}: \begin{cases} \dot{\hat{\phi}}_{[l]} = \mathrm{dg}(\hat{\mathbf{E}}_{[l]i})\hat{\phi}_{[l]} + \hat{\mathbf{\Gamma}}_{[l]}y + \sum_{k=l}^{L} \mathrm{dg}(\hat{\mathbf{\Lambda}}_{[l,k]i})_{i\in\mathbb{N}_{[k]}} u_{[k]} \\ u_{[l]} = \mathrm{dg}(\hat{\mathbf{F}}_{[l]i})\hat{\phi}_{[l]} + \hat{\mathbf{G}}_{[l+1]}\hat{\phi}_{[l+1]} \end{cases} \tag{13}$$

with $\hat{\phi}_{[l]}(0) = 0$ and

$$v = \mathrm{dg}(\hat{\mathbf{U}}_i)\hat{\phi}_{[1]}$$

where $\hat{\mathbf{G}}_{[L+1]}$ and $\hat{\phi}_{[L+1]}$ are regarded as zero. Furthermore,

$$\hat{\mathbf{E}}_{[l]i} \in \mathbb{R}^{\hat{n}_{[l]i}\times\hat{n}_{[l]i}}, \quad \hat{\mathbf{F}}_{[l]i} \in \mathbb{R}^{m_{[l]i}\times\hat{n}_{[l]i}}, \quad \hat{\mathbf{\Gamma}}_{[l]} \in \mathbb{R}^{\hat{n}_{[l]}\times p}$$
$$\hat{\mathbf{G}}_{[l+1]} \in \mathbb{R}^{m_{[l]}\times\hat{n}_{[l+1]}}, \quad \hat{\mathbf{U}}_i \in \mathbb{R}^{q_i\times\hat{n}_{[1]i}}$$

and $\hat{\mathbf{\Lambda}}_{[l,k]i}$, whose dimension is defined such that $\mathrm{dg}(\hat{\mathbf{\Lambda}}_{[l,k]i})_{i\in\mathbb{N}_{[k]}} \in \mathbb{R}^{\hat{n}_{[l]}\times m_{[k]}}$, are design parameters in conjunction with $\hat{n}_{[l]i}$ and $\hat{n}_{[l]}$ satisfying

$$\hat{n}_{[l]} = \sum_{i\in\mathbb{N}_{[l]}} \hat{n}_{[l]i}.$$

Note that $\hat{\mathbf{\Lambda}}_{[l,k]i}$ depends on not only k, but also l, unlike the case of $\mathbf{\Lambda}_{[k]i}$ in (8) because the input port of $\hat{\Phi}_{[l]}$ with respect to $u_{[k]}$ is different for each $l \in \mathbb{L}$ in general. In what follows, we deal with the case of $\hat{n}_{[l]i} \leq n_{[l]i}$.

Towards the distributed design of locally stabilizing controllers $\{\kappa_i\}$ in (2), we consider designing $\{\hat{\Phi}_{[l]}\}$ in (13) such that the $\mathcal{L}_2$-induced norm performance of the closed-loop system $((\Sigma, \{\hat{\Phi}_{[l]}\}), \{\kappa_i\})$ improves as that of the local closed-loop systems (10) improves. To this end, we take a controller reduction approach. Namely, we first suppose that a desirable $\{\Phi_{[l]}\}$ in (8) and (9) is given, e.g. it achieves a desirable $\gamma_{[l]}$ in (11). Next, we approximate the given $\{\Phi_{[l]}\}$ by $\{\hat{\Phi}_{[l]}\}$ such that the resultant trajectory of the state variables x of Σ in the closed-loop system $((\Sigma, \{\hat{\Phi}_{[l]}\}), \{\kappa_i\})$ gets closer to that in the closed-loop system $((\Sigma, \{\Phi_{[l]}\}), \{\kappa_i\})$ as long as the $\mathcal{L}_2$-induced norm performance of local closed-loop system (10) improves. In view of this, we state the following controller reduction problem:

Problem 1: Given $\{\mathbb{N}_{[l]}\}$ and $\{\mathbb{C}_{[l]i}\}_{i\in\mathbb{N}_{[l+1]}}$ as in (5) and (6), consider Σ in (7). Let $\{\Phi_{[l]}\}$ in (8) and (9) be given such that it achieves a desirable $\gamma_{[l]}$ in (11). Let $\hat{x} \in \mathbb{R}^n$ denote the state variables of Σ in the closed-loop system $((\Sigma, \{\hat{\Phi}_{[l]}\}), \{\kappa_i\})$. Given constant $\epsilon > 0$, find $\{\hat{\Phi}_{[l]}\}$ in (13) satisfying

$$\|x(t) - \hat{x}(t)\|_{\mathcal{L}_2} \leq \epsilon\|\chi(t)\|_{\mathcal{L}_2} \tag{14}$$

where χ is defined in (10) and $x(0) = \hat{x}(0) = x_0$ for all $x_0 \in \mathbb{R}^n$.

It should be noted that the difficulties of this problem are that

1) the error bound should improve as the $\mathcal{L}_2$-performance of individual local closed-loop system improves, and
2) the approximant $\{\hat{\Phi}_{[l]}\}$ has the hierarchical distributed structure in (13).

In the next section, we give a solution to Problem 1 by explicitly utilizing the structure of the closed-loop system.

III. Low-dimensional Hierarchical Distributed Compensator Design

A. Controller Reduction

In the rest of this paper, for simplicity, we focus on the case $L = 3$, which yields $\mathbb{L} = \{1, 2, 3\}$. We have a similar result in general cases. In addition, we omit the subscript 1 of the matrices associated with $\Phi_{[3]}$ and $\hat{\Phi}_{[3]}$, e.g., $E_{[3]}$ denotes $E_{[3]1}$.

Note that $\Phi_{[3]}$ and each compositional unit of $\Phi_{[2]}$, which is n-dimensional system and $n_{[2]i}$-dimensional system, respectively, are of higher dimension than $\Phi_{[1]i}$ (see also Fig. 2). In view of this, we consider reducing $\Phi_{[3]}$ and $\Phi_{[2]}$. In other words, we take $\hat{\Phi}_{[1]}$ as $\Phi_{[1]}$, i.e., the parameters of $\hat{\Phi}_{[1]}$ in (13) are taken as

$$\hat{\mathbf{E}}_{[1]i} = \mathbf{E}_{[1]i}, \quad \hat{\mathbf{F}}_{[1]i} = \mathbf{F}_{[1]i}, \quad \hat{\mathbf{\Gamma}}_{[1]} = \mathbf{\Gamma}_{[1]}$$
$$\hat{\mathbf{U}}_i = \mathbf{U}_i, \quad \hat{\mathbf{\Lambda}}_{[1,k]i} = \mathbf{\Lambda}_{[1]i} \tag{15}$$

for each $i \in \mathbb{N}_{[1]}$ and $k \in \mathbb{L}$.

To guarantee (14), it suffices that the approximation error of the closed-loop systems is independently evaluated by using individual local closed-loop systems (10) and systems without including any information on locally stabilizing controllers. More specifically, let us consider the following two transfer matrices associated with $\{\Phi_{[l]}\}$ and $\{\hat{\Phi}_{[l]}\}$:

$$g(s) := [0, I_n](sI - \mathcal{A})^{-1}\mathcal{B}, \quad \hat{g}(s) := [0, I_n](sI - \hat{\mathcal{A}})^{-1}\hat{\mathcal{B}} \tag{16}$$

where $\mathcal{A}$, $\hat{\mathcal{A}}$, $\mathcal{B}$ and $\hat{\mathcal{B}}$ are given in (17). Note that they are completely independent from locally stabilizing controllers $\{\kappa_i\}$ in (2). Using these systems, we evaluate the approximation error as follows:

Theorem 1: Consider Problem 1 and $\{\hat{\Phi}_{[l]}\}$ in (13) and (15). Define $g(s)$ and $\hat{g}(s)$ in (16). If $\hat{\mathcal{A}}$ is stable, then

$$\|x(t) - \hat{x}(t)\|_{\mathcal{L}_2} \leq \|\chi(t)\|_{\mathcal{L}_2}\,\|g(s) - \hat{g}(s)\|_{\mathcal{H}_\infty} \tag{18}$$

where χ is defined as in (10) and $x(0) = \hat{x}(0) = x_0$ for any $x_0 \in \mathbb{R}^n$.

Proof: Omit the proof due to page limitation. ■

Theorem 1 shows that we can solve Problem 1 by finding $\hat{\Phi}_{[2]}$ and $\hat{\Phi}_{[3]}$ such that $\|g(s) - \hat{g}(s)\|_{\mathcal{H}_\infty} < \epsilon$. We next clarify the relation between the approximation error of $\Phi_{[l]}$ and $\hat{\Phi}_{[l]}$, and that of $g(s)$ and $\hat{g}(s)$. More specifically, we take the

$$\mathcal{A} := \begin{bmatrix} \mathbf{E}_{[3]} + \mathbf{\Lambda}_{[3]}\mathbf{F}_{[3]} & 0 & \mathbf{\Gamma}_{[3]}\mathrm{dg}(C_i) \\ \mathbf{\Lambda}_{[3]}\mathbf{F}_{[3]} + \mathrm{dg}(\mathbf{\Lambda}_{[2]i})\mathbf{G}_{[3]} & \mathrm{dg}(\mathbf{E}_{[2]i} + \mathbf{\Lambda}_{[2]i}\mathbf{F}_{[2]i}) & \mathbf{\Gamma}_{[2]}\mathrm{dg}(C_i) \\ \mathbf{\Lambda}_{[3]}\mathbf{F}_{[3]} + \mathrm{dg}(\mathbf{\Lambda}_{[2]i})\mathbf{G}_{[3]} & \mathrm{dg}(\mathbf{\Lambda}_{[2]i}\mathbf{F}_{[2]i}) + \mathrm{dg}(\mathbf{\Lambda}_{[1]i})\mathbf{G}_{[2]} & A + \mathrm{dg}(\mathbf{\Lambda}_{[1]i}\mathbf{F}_{[1]i}) \end{bmatrix} \in \mathbb{R}^{3n\times 3n}$$

$$\hat{\mathcal{A}} := \begin{bmatrix} \hat{\mathbf{E}}_{[3]} + \hat{\mathbf{\Lambda}}_{[3,3]}\hat{\mathbf{F}}_{[3]} & 0 & \hat{\mathbf{\Gamma}}_{[3]}\mathrm{dg}(C_i) \\ \hat{\mathbf{\Lambda}}_{[2,3]}\hat{\mathbf{F}}_{[3]} + \mathrm{dg}(\hat{\mathbf{\Lambda}}_{[2,2]i})\hat{\mathbf{G}}_{[3]} & \mathrm{dg}(\hat{\mathbf{E}}_{[2]i} + \hat{\mathbf{\Lambda}}_{[2,2]i}\hat{\mathbf{F}}_{[2]i}) & \hat{\mathbf{\Gamma}}_{[2]}\mathrm{dg}(C_i) \\ \mathbf{\Lambda}_{[3]}\hat{\mathbf{F}}_{[3]} + \mathrm{dg}(\mathbf{\Lambda}_{[2]i})\hat{\mathbf{G}}_{[3]} & \mathrm{dg}(\mathbf{\Lambda}_{[2]i}\mathbf{F}_{[2]i}) + \mathrm{dg}(\mathbf{\Lambda}_{[1]i})\hat{\mathbf{G}}_{[2]} & A + \mathrm{dg}(\mathbf{\Lambda}_{[1]i}\mathbf{F}_{[1]i}) \end{bmatrix} \in \mathbb{R}^{(\hat{n}_{[3]}+\hat{n}_{[2]}+n)\times(\hat{n}_{[3]}+\hat{n}_{[2]}+n)}$$

$$\mathcal{B} := \begin{bmatrix} \mathbf{\Gamma}_{[3]}\mathrm{dg}(C_i) \\ \mathbf{\Gamma}_{[2]}\mathrm{dg}(C_i) \\ \mathbf{\Gamma}_{[1]}\mathrm{dg}(C_i) \end{bmatrix} \in \mathbb{R}^{3n\times n}, \quad \hat{\mathcal{B}} := \begin{bmatrix} \hat{\mathbf{\Gamma}}_{[3]}\mathrm{dg}(C_i) \\ \hat{\mathbf{\Gamma}}_{[2]}\mathrm{dg}(C_i) \\ \mathbf{\Gamma}_{[1]}\mathrm{dg}(C_i) \end{bmatrix} \in \mathbb{R}^{(\hat{n}_{[3]}+\hat{n}_{[2]}+n)\times n} \tag{17}$$

orthogonal projection [12], i.e., the parameters in (13) are taken as

$$\begin{aligned} &\hat{\mathbf{E}}_{[l]i} = \mathbf{P}_{[l]i}\mathbf{E}_{[l]i}\mathbf{P}_{[l]i}^{\mathsf{T}}, && \hat{\mathbf{F}}_{[l]i} = \mathbf{F}_{[l]i}\mathbf{P}_{[l]i}^{\mathsf{T}} \\ &\hat{\mathbf{\Gamma}}_{[l]} = \mathrm{dg}(\mathbf{P}_{[l]i})\mathbf{\Gamma}_{[l]}, && \hat{\mathbf{\Lambda}}_{[l,l]i} = \mathbf{P}_{[l]i}\mathbf{\Lambda}_{[l]i} \\ &\hat{\mathbf{G}}_{[l]} = -\mathrm{dg}(\mathbf{F}_{[l-1]i})\mathrm{dg}(\mathbf{P}_{[l]i}^{\mathsf{T}}), && \hat{\mathbf{\Lambda}}_{[2,3]} = \mathrm{dg}(\mathbf{P}_{[2]i})\mathbf{\Lambda}_{[3]} \end{aligned} \tag{19}$$

for $l \in \{2,3\}$ where $\mathbf{P}_{[l]i} \in \mathbb{R}^{\hat{n}_{[l]i}\times n_{[l]i}}$ satisfies

$$\mathbf{P}_{[l]i}\mathbf{P}_{[l]i}^{\mathsf{T}} = I_{\hat{n}_{[l]i}}. \tag{20}$$

In this formulation, Problem 1 coincides with the problem to find $\mathbf{P}_{[l]i}$ in (19) such that $\|g(s) - \hat{g}(s)\|_{\mathcal{H}_\infty} < \epsilon$, whose solution is given by the following theorem:

Theorem 2: Consider Problem 1 and $\{\hat{\Phi}_{[l]}\}$ in (13), (15) and (19). Define

$$\begin{aligned} \sigma_{[l,k]} := \|&(I - \mathrm{dg}(\mathbf{P}_{[l]i}^{\mathsf{T}}\mathbf{P}_{[l]i})) \\ &\times(sI - \mathrm{dg}(A_{[k]i} + B_{[k]i}\mathbf{F}_{[k]i}))^{-1}\mathrm{dg}(J_{[k]i})\mathrm{dg}(C_i)\|_{\mathcal{H}_\infty} \end{aligned} \tag{21}$$

for each $k \geq l$ and $l \in \{2,3\}$. Let $\mathcal{A}$ be given by (17) and

$$\mathcal{P} := \mathrm{dg}(\mathbf{P}_{[3]}, \mathrm{dg}(\mathbf{P}_{[2]i}), I_n) \tag{22}$$

If $\mathbf{P}_{[l]i}$ such that $\mathcal{P}\mathcal{A}\mathcal{P}^{\mathsf{T}}$ is stable, then

$$\|g(s) - \hat{g}(s)\|_{\mathcal{H}_\infty} \leq \sigma\mu \tag{23}$$

where $g(s)$ and $\hat{g}(s)$ are defined in (16) and

$$\begin{aligned} \sigma &:= ((\sigma_{[2,3]} + \sigma_{[3,3]})(1+\gamma_{[2]}) + \sigma_{[2,2]})(1+\gamma_{[1]}) \\ \mu &:= \|[0, I_n](sI - \mathcal{P}\mathcal{A}\mathcal{P}^{\mathsf{T}})^{-1}\mathcal{P}\mathcal{A}[I_{2n}, 0]^{\mathsf{T}}\|_{\mathcal{H}_\infty} \end{aligned} \tag{24}$$

with $\gamma_{[l]}$ defined in (11).

Proof: Omit the proof due to page limitation. ■

Theorem 2 shows that the approximation error $\|g(s) - \hat{g}(s)\|_{\mathcal{H}_\infty}$ is bounded by $\sigma_{[l,k]}$ associated with the approximation error of $\Phi_{[l]}$ and $\hat{\Phi}_{[l]}$. Thus, we can make the approximation error small if we find $\mathbf{P}_{[l]i}$ such that $\sigma_{[l,k]}$ is sufficiently small and $\mathcal{P}\mathcal{A}\mathcal{P}^{\mathsf{T}}$ is stable.

B. Design Procedure for Low-dimensional Hierarchical Distributed Compensator

In this subsection, we provide a design procedure for the low-dimensional hierarchical distributed compensators. In general, it is difficult to find an orthogonal projection to satisfy a criterion evaluated by the $\mathcal{H}_\infty$-norm, such as (21). Intuitively, we can expect that the $\mathcal{H}_\infty$-norm of a transfer matrix is small if the $\mathcal{H}_2$-norm of that is small (see [13] for sufficient conditions). In view of this, we determine $\mathbf{P}_{[l]i}$ such that the $\mathcal{H}_2$-norm of the system in the left side of (21) is small for a given $\hat{n}_{[l]i}$. Specific procedure is provided in [14]. Furthermore, if an approximation error is sufficiently small, then $\mathcal{P}\mathcal{A}\mathcal{P}^{\mathsf{T}}$ is expected to be stable. We summarize the design procedure for $\{\hat{\Phi}_{[l]}\}$ as follows:

1) Give $\epsilon > 0$.
2) For a given system Σ in (7) and a hierarchical structure $\{\mathbb{N}_{[l]}\}$ and $\{\mathbb{C}_{[l]i}\}_{i\in\mathbb{N}_{[l+1]}}$ for $l \in \{0,1,2\}$, construct $\{\Phi_{[l]}\}$ in (8) and (9) having desirable $\gamma_{[l]}$ in (11).
3) Find $\mathbf{P}_{[l]i}$ in (19) satisfying (20) and making $\sigma_{[l,k]}$ in (21) small for a given $\hat{n}_{[l]i}$.
4) If $\hat{g}(s)$ is unstable or $\|g(s) - \hat{g}(s)\|_{\mathcal{H}_\infty} > \epsilon$, then take a larger value of $\hat{n}_{[l]i}$ and go to step 3).
5) Construct $\{\hat{\Phi}_{[l]}\}$ in (13), (15) and (19).

IV. Numerical Example

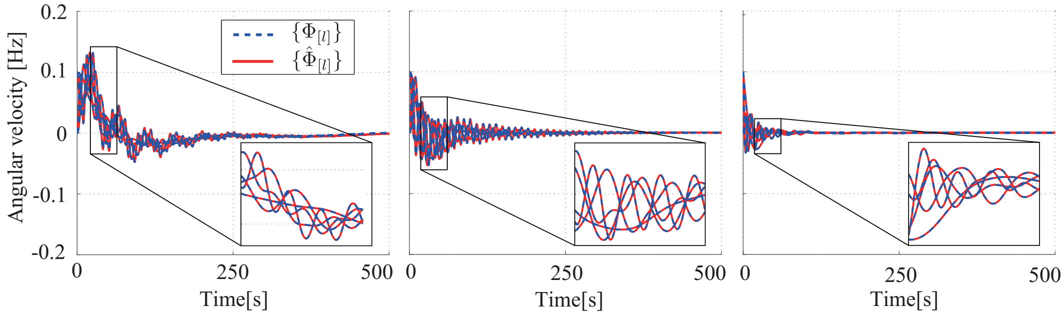
In this section, we show the efficiency of the proposed method by a power network example from [9]: The power network composed of $N = 50$ subsystems (areas) and each subsystem consists of three generators and two loads. Each generator and load are given as three- and two-dimensional systems, respectively. Thus, each subsystem is 13-dimensional, i.e., $n_i = 13$ for $i \in \mathbb{N} := \{1, \ldots, 50\}$, and the whole power networked system is 650-dimensional, i.e., $n = 650$.

The interconnection structure among generators and loads in each subsystem is given as a graph Laplacian called the Wattz-Strogatz(WS) model [15]. Furthermore, one generator in each subsystem connects to generators in the other subsystems, whose interconnection structure is given as a WS model.

Furthermore, the elements of the admittance matrix compatible with the subsystem interconnections are randomly chosen from $[0.1, 0.5]$, and those compatible with interconnection inside the subsystems are randomly chosen from the interval $[0.1, 1.0]$. In what follows, we consider a situation where the frequency of the power system suddenly varies. To simulate this, we give nonzero initial values for the angular velocity of the generators.

Next, we design a hierarchical distributed compensator $\{\Phi_{[l]}\}$ as follows: We take $L = 3$ and

$$|\mathbb{N}_{[1]}| = 10, \quad |\mathbb{N}_{[2]}| = 2, \quad |\mathbb{N}_{[3]}| = 1$$



(a) Low-gain local controllers (b) Middle-gain local controllers (c) High-gain local controllers

Fig. 3. Initial value responses of a power network model.

and take cluster sets $\mathbb{C}_{[l]i}$ having the same number of subsystems in each layer, i.e.,

$$|\mathbb{C}_{[0]i}| = 5, \quad |\mathbb{C}_{[1]i}| = 5, \quad |\mathbb{C}_{[2]i}| = 2$$

for $i \in \mathbb{N}_{[l]}$. Thus, we have $n_{[1]i}$ = 65, $n_{[2]i}$ = 325 and $n_{[3]}$ = 650 for $i \in \mathbb{N}_{[l]}$. Furthermore, let $m_{[1]}$ = 25, $m_{[2]}$ = 17 and $m_{[3]}$ = 13, which are the number of generators having additional input ports compatible with $B_{[l]}$, and take such generators randomly from 50 generators connecting to subsystems. For each $l \in \{1, 2, 3\}$, we design $\mathbf{F}_{[l]i}$ minimizing $\gamma_{[l]}$ in (11) and construct $\Phi_{[l]}$ by (8) and (9).

In Fig. 3, we depict the initial value responses of the closed-loop system with a hierarchical distributed compensator $\{\Phi_{[l]}\}$ by the blue dotted lines. From this figure, we see that the $\mathcal{L}_2$-induced norm performance of the closed-loop system improves as that of the local closed-loop system (10) improves. However, each compositional unit of the designed compensator $\Phi_{[2]}$ and $\Phi_{[3]}$ is a 325- and 650-dimensional system, respectively.

Next, we consider reducing those two compensators while preserving a similar performance. Let $\hat{\Phi}_{[l]}$ as $\hat{n}_{[2]i} = 200$ and $\hat{n}_{[3]} = 137$, the resultant approximation error is $\|g(s) - \hat{g}(s)\|_{\mathcal{H}_\infty}/\|g(s)\|_{\mathcal{H}_\infty} = 0.017$. In Fig. 3, we also plot the initial value responses of the closed-loop system with $\{\hat{\Phi}_{[l]}\}$ by the red solid lines. We can see from this figure that the trajectories compatible with $\hat{\Phi}_{[l]}$ are close to those compatible with $\Phi_{[l]}$. Furthermore, for each case (a)-(c), the resultant value of $\sup_{x_0}(\|x(t)\|_{\mathcal{L}_2}/\|x(0)\|_2)$ is 3568, 477 and 229, respectively. Therefore, we can see that the $\mathcal{L}_2$-induced norm performance of the closed-loop system improves as that of the performance of the local closed-loop system gets better.

V. Conclusion

In this paper, towards distributed design of locally stabilizing controllers, we have proposed a design method of low-dimensional hierarchical distributed compensators for large-scale networked linear systems. We have formulated the problem of designing such compensators as a structured controller reduction problem such that the approximation error bound improves as the performance of local closed-loop system improves. To solve this problem, we have shown that the approximation error of the closed-loop system can be evaluated by that of the system associated with the hierarchical distributed compensator without locally stabilizing controllers. On the basis of this result, we have derived the relation between the approximation errors of the individual hierarchical distributed compensators and the performance degradation of the closed-loop system. Finally, the efficiency of the proposed method has been demonstrated through a numerical example of a power network.

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