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Stability of State Estimation over Sensor Networks with Markovian Fading Channels ⋆

Daniel E. Quevedo * **Anders Ahlén** ** **Karl H. Johansson** ***

* *School of Electrical Engineering & Computer Science, The University of Newcastle, NSW 2308, Australia.* dquevedo@ieee.org
** *Department of Engineering Sciences, Signals and Systems, Uppsala University, Uppsala, Sweden.* Anders.Ahlen@signal.uu.se
*** *ACCESS Linnaeus Centre, School of Electrical Engineering, Royal Institute of Technology, Stockholm, Sweden.* kallej@ee.kth.se

Abstract: Stochastic stability for centralized Kalman filtering over a wireless sensor network with correlated fading channels is studied. On their route to the gateway, sensor packets, possibly aggregated with measurements from several nodes, may be dropped because of fading links. By assuming the network states to be Markovian, we establish sufficient conditions that ensure the Kalman filter to be exponentially bounded in norm. In the one sensor case, this new stability condition is shown to include previous results obtained in the literature as special cases. The results also hold when applying power control, where the transmission power of each node is a nonlinear mapping of the network state and the channel gains.

Keywords: Sensor networks; State estimation; Stochastic stability; Packet dropouts; Gilbert-Elliot model; Fading communication channel.

1. INTRODUCTION

Wireless sensor technology is of growing interesting for process and automation industry. The driving force behind using wireless technology in monitoring and control applications is its lower deployment and reconfiguration cost, e.g., Alur et al. (2009). In addition, wireless devices can be placed where wires cannot go, or where power sockets are not available, see Ilyas et al. (2004); Shen et al. (2007).

A drawback of wireless communication technology lies in that wireless channels are subject to fading and interference, which frequently lead to packet errors. The wireless channel is in general time varying. This time variability may in an industrial setting be caused by moving machines, vehicles, people, and so forth, or when the receiver or the transmitter are mounted on a moving object. Therefore, in addition to the propagation path loss, channels will commonly be subject to shadow and small scale fading, see Goldsmith (2005). The time-variability of the fading channel can be partially compensated for through control of the power levels used by the radio amplifiers. Several interesting approaches have been reported for state estimation of linear time-invariant (LTI) systems via wireless sensor networks. For example, the works Shi et al. (2010) and Shi (2009) focus on delay issues in a multiple-sensor network with no dropouts, whereas Gupta et al. (2009) studies the effect of dropouts within an architecture with only one sensor node, but where intermediate nodes are allowed to process data.

In the present work, we study centralized state estimation for LTI systems via wireless sensor networks. The fading channels introduce random packet dropouts. In our approach, we allow the channel gains to be correlated in time and between each other. We also account for power control of sensor radio amplifiers. Based on motivating case studies from process industry, we assume that in-network processing is much faster than the dynamics of the system whose state is being estimated and, thus, neglect delays introduced by the network. By using elements of stochastic stability theory, we derive sufficient conditions on the system and network parameters for the covariance matrix of the state estimation error to be exponentially bounded in norm. In special cases, the results obtained correspond to conditions which have been previously documented in the literature on estimation with packet dropouts.

The remainder of the paper is organized as follows: Section 2 describes the sensor network architecture and Section 3 the communication model adopted. Section 4 characterizes the associated state estimator. In Section 5, we derive the main technical result. Its proof is given in Section 7. Section 6 studies the special case where the network has only one sensor. Section 8 draws conclusions.

Notation: We write $\mathbb{N}$ for $\{1, 2, 3, \ldots\}$, and $\mathbb{N}_0$ for $\mathbb{N} \cup \{0\}$. The notation $\{\nu\}_{\mathbb{N}_0}$ refers to the sequence $\{\nu(0), \nu(1), \ldots\}$, and $\{\nu\}_0^k$ to $\{\nu(0), \nu(1), \ldots, \nu(k)\}$. The notation $|\cdot|$ refers to cardinality of a set. The trace of a matrix A is denoted by $\operatorname{tr} A$, and its norm by $||A|| \triangleq \sqrt{\max \operatorname{eigs}(A^T A)}$, where

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eigs(A^TA) are the eigenvalues of A^TA and the superscript T refers to transposition. If a matrix A is positive definite (non-negative definite), then we write $A \succ 0$ ($A \succeq 0$). To denote the conditional probability of an event Ω given Δ, we write $\mathbf{Pr}\{\Omega \,|\, \Delta\}$. The expected value of a random variable μ given Δ, is denoted via $\mathbf{E}\{\mu \,|\, \Delta\}$, whereas for the unconditional expectation we write $\mathbf{E}\{\mu\}$. A real random variable η, which is zero-mean Gaussian with covariance matrix Γ is denoted by $\eta \sim \mathcal{N}(0, \Gamma)$.

2. SENSOR NETWORK ARCHITECTURE

Consider an uncontrolled LTI n-dimensional system:

$$x(k+1) = Ax(k) + w(k), \quad k \in \mathbb{N}_0, \tag{1}$$

where $x(0) \sim \mathcal{N}(x_0, P_0)$, with $x_0^T x_0 < \infty$, $\|P_0\| < \infty$. The driving noise process $\{w\}_{\mathbb{N}_0}$ is independent and identically distributed (i.i.d.), with $w(k) \sim \mathcal{N}(0, Q)$, $\forall k \in \mathbb{N}_0$.

To estimate the system state sequence $\{x\}_{\mathbb{N}_0}$, a collection of M wireless sensors $\{S_1, \dots, S_M\}$ is used. Each sensor provides a noisy measurement sequence $\{y_m\}_{\mathbb{N}_0}$ of the form

$$y_m(k) = C_m x(k) + v_m(k),\ C_m \in \mathbb{R}^{l_m \times n},\ m \in \{1, \dots, M\}, \tag{2}$$

with $C_m \in \mathbb{R}^{l_m \times n}$, $l_m \in \mathbb{N}$. In (2), the measurement noise processes $\{v_m\}_{\mathbb{N}_0}$ are i.i.d., with each $v_m(k) \sim \mathcal{N}(0, R_m)$.

The M (possibly vector) measurements in (2) are to be transmitted via wireless links to a single gateway (or fusion centre), denoted S_0. Since the links are wireless, unavoidably some measurements will be dropped by the network. The received measurement values are used to remotely estimate the state of the system (1).

We will assume that the network is much faster than the process (1) and will therefore neglect any delays experienced by the data when traveling through the network. Each sensor node aggregates its own measurements to the received packets from incoming nodes and transmits the resulting packet to a single destination node. Sensor nodes do not buffer old data. Thus, the measurements received by the gateway at time k are a subset of $\{y_1(k), y_2(k), \dots, y_M(k)\}$.

It is convenient to describe the network by means of a graph, with vertices $\{S_0, \dots S_M\}$, and edges associated with the wireless links. As noted before, each sensor S_m transmits to a single node, called its *parent* and henceforth denoted via $\mathrm{Par}(S_m)$. Thus, the graph constitutes a directed tree graph with root S_0. Each sensor node S_m has a single outgoing edge, say,

$$\mathcal{E}_m = \big(S_m, \mathrm{Par}(S_m)\big) \in \mathbb{E} \tag{3}$$

where

$$\mathbb{E} \triangleq \{\mathcal{E}_1, \mathcal{E}_2, \dots, \mathcal{E}_M\} \tag{4}$$

denotes the set of all edges of the tree. Furthermore, there exists a unique path from each S_m to the gateway. We denote this path by $\mathrm{Path}(S_m)$, its edges by $\mathrm{Edge}(\mathrm{Path}(S_m))$ and its nodes by $\mathrm{Node}(\mathrm{Path}(S_m))$.

Example 1. In Fig. 1, the packet transmitted by S_3 at time $k \in \mathbb{N}_0$ contains $y_3(k)$ and a subset of $\{y_6(k), y_7(k)\}$. We also have $\mathrm{Par}(S_4) = \mathrm{Par}(S_5) = S_2$, $\mathrm{Node}(\mathrm{Path}(S_7)) = \{S_7, S_3, S_1, S_0\}$, and $\mathrm{Edge}(\mathrm{Path}(S_7)) = \{\mathcal{E}_7, \mathcal{E}_3, \mathcal{E}_1\} = \{(S_7, S_3), (S_3, S_1), (S_1, S_0)\}$. □

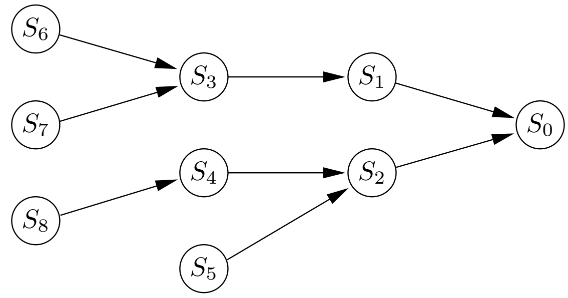

Fig. 1. Sensor network tree with 9 nodes and 8 edges.

3. TRANSMISSION EFFECTS

Since the links used to convey measurements from the sensors to the gateway are wireless, transmission errors are likely to occur. We will model transmission effects via random packet dropouts at the individual links of the network and, hence, introduce the binary stochastic communication success processes $\{\gamma_{\mathcal{E}}\}_{\mathbb{N}_0}$, $\mathcal{E} \in \mathbb{E}$, where:

$$\gamma_{\mathcal{E}}(k) = \begin{cases} 1 & \text{if at time } k \\ & \quad \text{transmission via } \mathcal{E} \in \mathbb{E} \text{ is successful,} \\ 0 & \text{otherwise.} \end{cases} \tag{5}$$

The distributions of the the processes $\{\gamma_{\mathcal{E}}\}_{\mathbb{N}_0}$ are determined by power levels and channel gains. More precisely, for each link $\mathcal{E}_m = (S_m, \mathrm{Par}(S_m)) \in \mathbb{E}$, we will write (with a slight abuse of notation):

$$\mathbf{Pr}\big\{\gamma_{\mathcal{E}_m}(k) = 1 \,\big|\, u_m(k), h_{\mathcal{E}_m}(k)\big\} = f_{\mathcal{E}_m}\big(u_m(k) h_{\mathcal{E}_m}(k)\big), \tag{6}$$

where $u_m(k)$ denotes the power used by the radio power amplifier of sensor S_m at time k, $h_{\mathcal{E}_m}(k)$ is the channel (power) gain from S_m to its parent node, and $f_{\mathcal{E}_m}(\cdot)\colon [0, \infty) \to [0, 1]$ is a monotonically increasing function, which depends on the modulation used; see, e.g., Proakis (1995).

Channel gains are modelled as random variables, which are affected by shadow and small scale fading; see Goldsmith (2005). Shadow fading is caused by large (and possibly slowly moving) objects obstructing the radio link, and can therefore be correlated in time and space. Thus, if sensors are close to each other, then shadow fading may cause correlations between the individual link gains. Small scale fading is commonly modelled via uncorrelated channel gain distributions.

To model shadow fading we introduce an overall discrete *network state* process $\{\Xi\}_{\mathbb{N}_0}$, where

$$\Xi(k) \in \mathbb{B} \triangleq \{0, 1, \dots, |\mathbb{B}| - 1\},$$

where $|\mathbb{B}|$ denotes the cardinality of the finite set $\mathbb{B}$. The network state process describes a finite number of configurations of the overall physical environment and, thus, governs the channel gains, which can be spatially correlated. To incorporate temporal correlations, we will allow the network states $\{\Xi\}_{\mathbb{N}_0}$ to form a discrete homogeneous Markov chain with transition probabilities

$$G(j, i) = \mathbf{Pr}\big\{\Xi(k+1) = j \,\big|\, \Xi(k) = i\big\}, \quad \forall i, j \in \mathbb{B},\ k \in \mathbb{N}_0, \tag{7}$$

see, e.g., Brémaud (1999).

Due to small-scale fading, for a given network state, channel gains are random and independent in time and of each other. More formally, if $\mathcal{E}_l \neq \mathcal{E}_m$ or $k \neq \ell$, then the channel gain distributions satisfy

$$\begin{aligned}\mathbf{Pr}\big\{h_{\mathcal{E}_l}(k) \leq h_1, h_{\mathcal{E}_m}(\ell) \leq h_2 \,\big|\, \Xi(k) = j, \Xi(\ell) = i\big\} \\ = \mathbf{Pr}\{h_{\mathcal{E}_l}(k) \leq h_1 \,|\, \Xi(k) = j\} \\ \times \mathbf{Pr}\{h_{\mathcal{E}_m}(\ell) \leq h_2 \,|\, \Xi(\ell) = i\},\end{aligned} \tag{8}$$

for all $h_1, h_2 \in \mathbb{R}$ and all $i, j \in \mathbb{B}$.

In view of (6), power control can be used to counteract fading effects; see Pantazis and Vergados (2007); Quevedo et al. (2010, 2011). In the present work, we consider power control laws to be of the form:

$$u_m(k) = \kappa_m(\Xi(k), h_{\mathcal{E}_m}(k)), \tag{9}$$

where $\kappa_m(\cdot,\cdot)$ are non-linear mappings. Particular cases include the use of fixed power levels and fixed gain controllers with saturated outputs, $u_m(k) = \mathrm{sat}\big(\mathcal{K}_m/h_{\mathcal{E}_m}(k)\big)$.

It is important to emphasize that the network state determines the distribution of the channel gains and, thereby, the distribution of the link success probabilities in (5). The fading model adopted is related to the Gilbert-Elliot Model, see Gilbert (1960). A distinguishing feature of our model is that in all network states Ξ the individual link success probabilities are described by probability distributions, see (8) and (9). Thus, whilst $\{\Xi\}_{\mathbb{N}_0}$ is assumed Markovian, we do not require that the channel gains $\{h_{\mathcal{E}}\}_{\mathbb{N}_0}$ or the dropout processes $\{\gamma_{\mathcal{E}}\}_{\mathbb{N}_0}$ be Markovian. We note, however, that i.i.d. transmissions and also Markovian models for the dropout processes, as studied in Xie and Xie (2008); Huang and Dey (2007); Smith and Seiler (2003), are special cases of our model.

A key feature of the network and power control model presented above is that, *when conditioned upon the network states* Ξ, the link transmission success processes are independent in time and of each other. For further reference, we will denote the associated success probabilities via

$$\phi_{\mathcal{E}_m}(j) \triangleq \mathbf{Pr}\big\{\gamma_{\mathcal{E}_m}(k) = 1 \,\big|\, \Xi(k) = j\big\}, \quad \mathcal{E}_m \in \mathbb{E},\ j \in \mathbb{B} \tag{10}$$

and note that:

$$\phi_{\mathcal{E}_m}(j) = \mathbf{E}\big\{f_{\mathcal{E}_m}\big(h_{\mathcal{E}_m} \cdot \kappa_m(\Xi, h_{\mathcal{E}_m})\big) \,\big|\, \Xi = j\big\}, \tag{11}$$

where $\mathcal{E}_m = (S_m, \mathrm{Par}(S_m))$. Thus, for given control policies $\kappa_m(\cdot,\cdot)$, calculating $\phi_{\mathcal{E}}(j)$ involves simply taking expectation with respect to the conditional distribution of the channel gain $h_{\mathcal{E}}$ given the network state $\Xi = j$.

Example 2. Suppose that the M links are independent of each other and that each link can be either operating "normally", or be in "outage", e.g., due to blocking by a large object. In this situation it is convenient to choose the network state as

$$\Xi(k) = \sum_{m=1}^{M} 2^{m-1}\Xi_{\mathcal{E}_m}(k) \tag{12}$$

where

$$\Xi_{\mathcal{E}}(k) \in \{0, 1\}, \quad \forall \mathcal{E} \in \mathbb{E}, \tag{13}$$

so that the set $\mathbb{B}$ becomes $\{0, 1, \ldots, 2^M - 1\}$. In (13), the value $\Xi_{\mathcal{E}}(k) = 1$ indicates that the link $\mathcal{E}$ is in normal operation, whereas $\Xi_{\mathcal{E}}(k) = 0$ models outage. Transition between the link states is random and obeys, see Fig. 2

$$\begin{aligned}\mathbf{Pr}\{\Xi_{\mathcal{E}}(k+1) = 1 \,|\, \Xi_{\mathcal{E}}(k) = 1\} &= 1 - p_{\mathcal{E}} \\ \mathbf{Pr}\{\Xi_{\mathcal{E}}(k+1) = 0 \,|\, \Xi_{\mathcal{E}}(k) = 1\} &= p_{\mathcal{E}} \\ \mathbf{Pr}\{\Xi_{\mathcal{E}}(k+1) = 0 \,|\, \Xi_{\mathcal{E}}(k) = 0\} &= 1 - q_{\mathcal{E}} \\ \mathbf{Pr}\{\Xi_{\mathcal{E}}(k+1) = 1 \,|\, \Xi_{\mathcal{E}}(k) = 0\} &= q_{\mathcal{E}},\end{aligned} \tag{14}$$

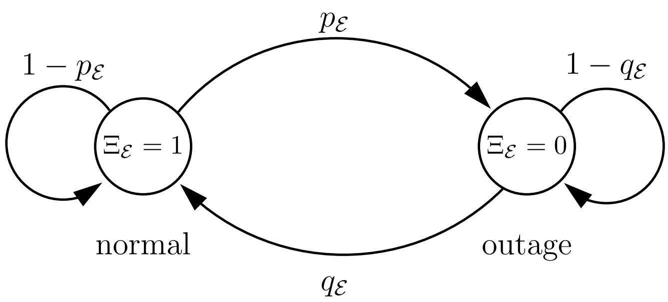

Fig. 2. Markovian model for links which are independent of each other, as described in Example 2.

where $p_{\mathcal{E}}$ is the *failure rate* and $q_{\mathcal{E}}$ is the *recovery rate*. □

4. STATE ESTIMATION OVER A SENSOR NETWORK TREE WITH PACKET DROP-OUTS

The purpose of the sensor network architecture considered is to estimate the state of the system (1) centrally at the gateway by using the measurements received from the sensors $\{S_1, S_2, \ldots, S_M\}$. As we have seen in Section 3, fading channels will introduce random packet loss. From an estimation point of view, it is convenient to introduce the binary processes $\{\theta_m\}_{\mathbb{N}_0}$, where:

$$\theta_m(k) = \begin{cases} 1 & \text{if at time } k \text{ transmission} \\ & \quad \text{via Path}(S_m) \text{ is successful,} \\ 0 & \text{otherwise.}\end{cases} \tag{15}$$

We will assume that the packets transmitted from the sensors to the gateway incorporate error detection coding, see, e.g., Proakis (1995), and that the gateway knows, whether received packets are correct or not. Thus, the information available for state estimation at the gateway at time k is given by

$$\mathcal{I}(k) = \Big\{\{\theta_1\}_0^k, \ldots, \{\theta_M\}_0^k, \{y\}_0^k\Big\}, \tag{16}$$

where

$$y(k) \triangleq \begin{bmatrix}\theta_1(k)y_1(k) \\ \theta_2(k)y_2(k) \\ \vdots \\ \theta_M(k)y_M(k)\end{bmatrix}, \quad k \in \mathbb{N}_0.$$

An important observation is that with power control laws of the form (9) and given the channel model adopted, the dropout realizations in (16) do not convey information about the system state $\{x\}_{\mathbb{N}_0}$. Since we have assumed that the network does not introduce any delays, it turns out that state estimation in the wireless sensor network configuration studied amounts to sampling the system (1) using the time-varying (stochastic) observation matrix

$$C(k) \triangleq \begin{bmatrix}\theta_1(k)C_1 \\ \theta_2(k)C_2 \\ \vdots \\ \theta_M(k)C_M\end{bmatrix}, \quad k \in \mathbb{N}_0. \tag{17}$$

Consequently, the conditional distribution of $x(k)$ given $\mathcal{I}(k-1)$ is Gaussian. The conditional mean of $x(k)$

$$\hat{x}(k|k-1) \triangleq \mathbf{E}\big\{x(k) \,\big|\, \mathcal{I}(k-1)\big\}$$

and the associated conditional error covariance matrix,

$$P(k \,|\, k-1) \triangleq \mathbf{E}\big\{\epsilon(k)\epsilon(k)^T \,\big|\, \mathcal{I}(k-1)\big\}$$

with $\epsilon(k) \triangleq x(k) - \hat{x}(k|k-1)$, satisfy the Kalman filter recursions (see, e.g., Anderson and Moore (1979)):

$$\begin{aligned}\hat{x}(k+1|k) &= A\hat{x}(k|k-1) + K(k)\big(y(k) - C(k)\hat{x}(k|k-1)\big)\\ P(k+1|k) &= AP(k|k-1)A^T + Q \qquad (18)\\ &\quad - K(k)C(k)P(k|k-1)A^T\end{aligned}$$

where $R \triangleq \mathrm{diag}\big(R_1, R_2, \ldots, R_M\big)$,

$K(k) \triangleq AP(k|k-1)C(k)^T\big(C(k)P(k|k-1)C(k)^T + R\big)^{-1}$,

and with initial values $P(0|-1) = P_0$ and $\hat{x}(0|-1) = x_0$.

Remark 3. A key difference of our approach when compared to that in Shi et al. (2010); Shi (2009), is that we consider packet dropouts. Thus, the estimation error covariance matrix will not be stationary. □

Remark 4. It is easy to see that, since we assume that the network does not introduce any delays, we have:

$$\theta_m(k) = \prod_{\mathcal{E}\in\mathrm{Edge}(\mathrm{Path}(S_m))} \gamma_{\mathcal{E}}(k), \quad \forall m \in \{1, \ldots, M\}.$$

Furthermore, the conditional distributions of $\{\theta_m\}$ given the network states can be written in terms of the individual link functions $\phi_{\mathcal{E}}(j)$:

$$\begin{aligned}&\mathbf{Pr}\{\theta_m(k) = 1 \,\big|\, \Xi(k) = j\}\\ &= \prod_{\mathcal{E}\in\mathrm{Edge}(\mathrm{Path}(S_m))} \mathbf{Pr}\{\gamma_{\mathcal{E}}(k) = 1 \,\big|\, \Xi(k) = j\}\\ &= \prod_{\mathcal{E}\in\mathrm{Edge}(\mathrm{Path}(S_m))} \phi_{\mathcal{E}}(j).\end{aligned}$$

Note, however, that if $S_m \in \mathrm{Node}(\mathrm{Path}(S_l))$, with $m \neq l$, then, in general,

$$\begin{aligned}&\mathbf{Pr}\{\theta_m(k) = 1, \theta_l(k) = 1 \,|\, \Xi(k) = j\}\\ &\neq \mathbf{Pr}\{\theta_m(k) = 1 \,|\, \Xi(k) = j\}\\ &\qquad \times \mathbf{Pr}\{\theta_l(k) = 1 \,|\, \Xi(k) = j\},\end{aligned}$$

compare to (8). □

5. STABILITY ANALYSIS

Due to packet dropouts, the covariance matrix $P(k+1|k)$ in (18) will, in general not converge to a fixed value and may, at times, diverge. As shown in Schenato et al. (2007), this type of behaviour occurs even in the simplest scenario, where only one sensor is used and dropout probabilities are i.i.d. We will next study stability of the Kalman filter (18) for the sensor network model at hand. For that purpose we will adopt the following stochastic stability notion:

Definition 5. The process $\{P(k+1|k)\}_{k\in\mathbb{N}_0}$ is said to be exponentially bounded in norm, if there exist finite constants α and β and $\rho \in [0, 1)$ such that:

$$\mathbf{E}\big\{\|P(k+1|k)\|\big\} \leq \alpha\rho^k + \beta, \quad \forall k \in \mathbb{N}_0. \qquad (19)$$

Accordingly, we say that the Kalman filter (18) is exponentially bounded. □

Our analysis makes use of the process $\{r\}_{\mathbb{N}_0}$, where:

$$r(k) = \begin{cases} 1 & \text{if } C(k) \text{ has full column-rank,}\\ 0 & \text{otherwise.}\end{cases}$$

Clearly, $r(k)$ is a (Boolean) function of the individual link success outcomes $\gamma_{\mathcal{E}}(k)$, $\mathcal{E} \in \mathbb{E}$. By the discussion in Section 3, it is easy to see that $r(k)$ is temporarily independent, when conditioned upon the network state $\Xi(k)$. We can therefore define

$$\phi_r(j) \triangleq \mathbf{Pr}\{r(k) = 1 \,|\, \Xi(k) = j\}. \qquad (20)$$

Note that $\phi_r(j)$ can be written in terms of the functions $\phi_{\mathcal{E}}(j)$ introduced in (10), see also Example 7 included at the end of this section.

The following theorem gives a sufficient condition for exponential stability of the Kalman filter used for state estimation over a sensor network with Markovian channel states:

Theorem 6. Define

$$\nu(i) \triangleq \sum_{j\in\mathbb{B}} (1 - \phi_r(j))G(j, i), \quad i \in \mathbb{B}, \qquad (21)$$

where $G(j, i)$ are the transition probabilities in (7). [1]

If there exists $\rho \in [0, 1)$ such that

$$\|A\|^2 \max_{i\in\mathbb{B}} \nu(i) \leq \rho, \qquad (22)$$

then the Kalman filter with the channel gain and power control model described in Section 3 is exponentially bounded in norm.

Proof. See Section 7. □

Our result establishes a sufficient conditions for a specific form of stochastic stability of the covariance matrix in (18), when the dropout process is governed by the model described in Section 3. The condition is stated in terms of a bound which involves the norm of the system matrix A, the transition probabilities of the channel state Ξ, and the conditional probabilities $\phi_r(j)$. The latter is determined by the individual conditional transmission success probabilities $\phi_{\mathcal{E}}(j)$, and can therefore be influenced by designing the power control policies, see (11). The situation generalizes that investigated for the simpler case of having independent channel gains in our recent work Quevedo et al. (2011).

Before turning our attention to a particular case, namely when the network has only one sensor, we will first give an example on how to calculate the functions $\phi_r(j)$.

Example 7. Consider the subgraph of the sensor network depicted in Fig. 1, having vertices $\{S_0, S_1, \ldots, S_5\}$. Suppose that for $C(k)$ to be of full-column rank (at least) three of the measurements $\{y_1(k), y_2(k), \ldots, y_5(k)\}$ need to be received at the gateway. Then $r(k) = 1$ if and only if

$$[\gamma_{\mathcal{E}_1}(k)\ \gamma_{\mathcal{E}_2}(k)\ \ldots\ \gamma_{\mathcal{E}_5}(k)]^T \in \mathbb{J},$$

where

$$\mathbb{J} = \left\{ \begin{bmatrix}0\\1\\0\\1\\1\end{bmatrix}, \begin{bmatrix}0\\1\\1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\1\\0\\0\\1\end{bmatrix}, \begin{bmatrix}1\\1\\0\\1\\0\end{bmatrix}, \begin{bmatrix}1\\1\\0\\1\\1\end{bmatrix}, \begin{bmatrix}1\\1\\1\\0\\0\end{bmatrix}, \begin{bmatrix}1\\1\\1\\0\\1\end{bmatrix}, \begin{bmatrix}1\\1\\1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\1\\1\\1\\1\end{bmatrix} \right\}$$

Since, as noted in Section 3, the link transmission success processes are conditionally independent, we obtain

[1] The term $\nu(i)$ denotes the total probability of $C(k+1)$ not having full column-rank, if the network state at time k is $\Xi(k) = i$.

$$\begin{aligned}\phi_r(j) = &(1-\phi_{\mathcal{E}_1}(j))\phi_{\mathcal{E}_2}(j)(1-\phi_{\mathcal{E}_3}(j))\phi_{\mathcal{E}_4}(j)\phi_{\mathcal{E}_5}(j)\\ &+(1-\phi_{\mathcal{E}_1}(j))\phi_{\mathcal{E}_2}(j)\phi_{\mathcal{E}_3}(j)\phi_{\mathcal{E}_4}(j)\phi_{\mathcal{E}_5}(j)\\ &+\phi_{\mathcal{E}_1}(j)\phi_{\mathcal{E}_2}(j)(1-\phi_{\mathcal{E}_3}(j))(1-\phi_{\mathcal{E}_4}(j))\phi_{\mathcal{E}_5}(j)\\ &+\phi_{\mathcal{E}_1}(j)\phi_{\mathcal{E}_2}(j)(1-\phi_{\mathcal{E}_3}(j))\phi_{\mathcal{E}_4}(j)(1-\phi_{\mathcal{E}_5}(j))\\ &+\phi_{\mathcal{E}_1}(j)\phi_{\mathcal{E}_2}(j)(1-\phi_{\mathcal{E}_3}(j))\phi_{\mathcal{E}_4}(j)\phi_{\mathcal{E}_5}(j)\\ &+\phi_{\mathcal{E}_1}(j)\phi_{\mathcal{E}_2}(j)\phi_{\mathcal{E}_3}(j)(1-\phi_{\mathcal{E}_4}(j))(1-\phi_{\mathcal{E}_5}(j))\\ &+\phi_{\mathcal{E}_1}(j)\phi_{\mathcal{E}_2}(j)\phi_{\mathcal{E}_3}(j)(1-\phi_{\mathcal{E}_4}(j))\phi_{\mathcal{E}_5}(j)\\ &+\phi_{\mathcal{E}_1}(j)\phi_{\mathcal{E}_2}(j)\phi_{\mathcal{E}_3}(j)\phi_{\mathcal{E}_4}(j)(1-\phi_{\mathcal{E}_5}(j))\\ &+\phi_{\mathcal{E}_1}(j)\phi_{\mathcal{E}_2}(j)\phi_{\mathcal{E}_3}(j)\phi_{\mathcal{E}_4}(j)\phi_{\mathcal{E}_5}(j),\end{aligned}$$

where $j \in \mathbb{B}$. □

6. THE ONE-SENSOR CASE

If the sensor network has only one sensor and one edge, namely S_1 and $\mathcal{E}_1$, then

$$\begin{aligned}P(k+1|k) &= AP(k|k-1)A^T + Q\\ &\quad - \gamma_{\mathcal{E}_1}(k)K(k)P(k|k-1)A^T \qquad (23)\\ K(k) &= AP(k|k-1)C_1^T\big(C_1P(k|k-1)C_1^T+R\big)^{-1}C_1.\end{aligned}$$

The question of stability of the Kalman filter when the packet dropout process $\{\gamma_{\mathcal{E}_1}\}_{\mathbb{N}_0}$ is i.i.d. has been extensively studied; see, e.g., Schenato et al. (2007). The case of Markovian packet dropout processes $\{\gamma_{\mathcal{E}_1}\}_{\mathbb{N}_0}$ was investigated in Xie and Xie (2008); Huang and Dey (2007). Corollary 8, given below, establishes exponential boundedness of the Kalman filter for the more general channel model presented in Example 2 of Section 3. [2]

Corollary 8. Consider the model of Example 2 in the one-sensor case. Suppose that C_1 in (2) has full column-rank and define:

$$\begin{aligned}\nu(0) &\triangleq \big(1-\phi_{\mathcal{E}_1}(0)\big)(1-q_{\mathcal{E}_1}) + \big(1-\phi_{\mathcal{E}_1}(1)\big)q_{\mathcal{E}_1}\\ \nu(1) &\triangleq \big(1-\phi_{\mathcal{E}_1}(0)\big)p_{\mathcal{E}_1} + \big(1-\phi_{\mathcal{E}_1}(1)\big)(1-p_{\mathcal{E}_1}).\end{aligned} \qquad (24)$$

If there exists $\rho \in [0,1)$ such that

$$\max\big(\nu(0),\nu(1)\big)\|A\|^2 \le \rho, \qquad (25)$$

then the Kalman filter is exponentially bounded in norm.

Proof. Follows directly from Theorem 6. □

If we now further restrict our setting, then Corollary 8 reduces to well-known results. To be more specific, a particular instance of the situation examined in Corollary 8 results when fixing $\phi_{\mathcal{E}_1}(0) = 0$ and $\phi_{\mathcal{E}_1}(1) = 1$. In this case, the dropout process $\{\gamma_{\mathcal{E}_1}\}_{\mathbb{N}_0}$ is Markovian, and the sufficient condition (25) becomes

$$\max(1-q_{\varepsilon_1}, p_{\varepsilon_1})\|A\|^2 \le \rho < 1$$

Perhaps not surprisingly, Corollary 8 then becomes akin to Theorem 3 in Xie and Xie (2008).

An even simpler situation results if, in (14), we set

$$p_{\mathcal{E}_1} = 1 - q_{\mathcal{E}_1} = 0,$$

i.e., success probabilities are independent. In this case, we have $\nu(0) = \nu(1) = 1 - \phi_{\mathcal{E}_1}(1)$ and (25) becomes

$$\big(1-\phi_{\mathcal{E}_1}(1)\big)\|A\|^2 \le \rho < 1,$$

thus, resembling various conditions which have been reported in the literature; see Schenato et al. (2007).

[2] We recall that in our model the link state $\{\Xi_{\mathcal{E}_1}\}_{\mathbb{N}_0}$ is Markovian, but $\{\gamma_{\mathcal{E}_1}\}_{\mathbb{N}_0}$ is, in general, not Markovian.

7. PROOF OF THEOREM 6

We first prepare two preliminary lemmas:

Lemma 9. The composite process $\{Z\}_{\mathbb{N}_0}$ defined via

$$Z(k) \triangleq \big(P(k|k-1), \Xi(k-1)\big), \quad k \in \mathbb{N}_0 \qquad (26)$$

is a Markov Chain.

Proof. Recall that the network state $\{\Xi\}_{\mathbb{N}_0}$ is Markovian, that for given network states the dropout processes are independent and that the power control policies satisfy (9). Therefore, the distribution of the matrix $C(k)$ satisfies

$$\begin{aligned}&\mathbf{Pr}\{C(k)=c\,|\,\Xi(k-1)=\Xi_{k-1},\Xi(k-2)=\Xi_{k-2},\dots\}\\ &\qquad = \mathbf{Pr}\{C(k)=c\,|\,\Xi(k-1)=\Xi_{k-1}\}.\end{aligned}$$

The result now follows from (18). □

Lemma 10. Define

$$V_k \triangleq \operatorname{tr} P(k|k-1). \qquad (27)$$

Then $V_k \ge 0$, for all $k \in \mathbb{N}_0$. Furthermore, there exists $W < \infty$, such that

$$\mathbf{E}\big\{V_1\,|\,Z(0)=(P_0,i)\big\} \le W + \nu(i)\big(\|A\|^2\operatorname{tr}P_0 + \operatorname{tr}Q\big), \qquad (28)$$

where $Z(0)$ is as defined in (26).

Proof. The fact that V_k is non-negative follows directly from $P(k|k-1)$ being a covariance matrix and, thus, non-negative definite. To prove (28), it is convenient to condition as follows:

$$\begin{aligned}&\mathbf{E}\big\{V_1\,|\,Z(0)=Z\big\}\\ &\quad = \mathbf{E}\big\{V_1\,|\,Z(0)=Z, r(0)=1\big\}\mathbf{Pr}\{r(0)=1\,|\,Z(0)=Z\}\\ &\quad\ + \mathbf{E}\big\{V_1\,|\,Z(0)=Z, r(0)=0\big\}\mathbf{Pr}\{r(0)=0\,|\,Z(0)=Z\}.\end{aligned} \qquad (29)$$

We next examine the outcomes $r(0) \in \{0,1\}$ separately.

1) For $r(0) = 1$, $C(0)$ has full column-rank. Therefore, a simple predictor for $x(1)$ given $y(0) \subset \mathcal{I}(0)$, is given by

$$\check{x}(1) = A\big(C(0)^TC(0)\big)^{-1}C(0)^Ty(0),$$

in which case

$$\check{x}(1) - x(1) = A\big(C(0)^TC(0)\big)^{-1}C(0)^Tv(0) - w(0).$$

Hence, there exists a constant $W < \infty$, such that

$$\mathbf{E}\big\{\big(\check{x}(1)-x(1)\big)\big(\check{x}(1)-x(1)\big)^T\big\} \preceq (W/n)I_n,$$

where I_n denotes the $n \times n$ identity matrix. Since the Kalman filter gives the minimum conditional error covariance matrix, and by the fact that for any square matrix F, $\mathbf{E}\{\operatorname{tr}F\} = \operatorname{tr}\mathbf{E}\{F\}$, we obtain the bound [3]

$$\begin{aligned}&\mathbf{E}\big\{V_1\,|\,Z(0)=Z, r(0)=1\big\}\mathbf{Pr}\{r(0)=1|Z(0)=Z\}\\ &\qquad \le W\mathbf{Pr}\{r(0)=1|Z(0)=Z\} \le W.\end{aligned} \qquad (30)$$

2) For the cases where $r(0) = 0$, the covariance matrix $P(1|0)$ is upper-bounded by that resulting form the worst case, where $\gamma_{\mathcal{E}}(0) = 0, \forall\mathcal{E} \in \mathbb{E}$. We, thus have:

$$\begin{aligned}&\mathbf{E}\big\{V_1\,|\,Z(0)=Z, r(0)=0\big\}\\ &\qquad \le \mathbf{E}\big\{V_1\,|\,Z(0)=Z, \gamma_{\mathcal{E}}(0)=0, \forall\mathcal{E}\in\mathbb{E}\big\}\\ &\qquad = \operatorname{tr}\{AP_0A^T+Q\} = \operatorname{tr}\{A^TAP_0\} + \operatorname{tr}Q\\ &\qquad \le \|A\|^2\operatorname{tr}P_0 + \operatorname{tr}Q,\end{aligned} \qquad (31)$$

where we have used (18) and (Bernstein, 2009, Fact 8.12.29).

[3] Clearly, (30) is not a tight bound, but it suffices for our purpose.

To calculate $\mathbf{Pr}\{r(0) = 0 \,|\, Z(0) = Z\}$, we condition upon $\Xi(0)$ and use the channel model to obtain:

$$\begin{aligned}
&\mathbf{Pr}\{r(0) = 0 \,|\, Z(0) = Z\} = \mathbf{Pr}\{r(0) = 0 \,|\, Z(0) = (P_0, i)\} \\
&= \sum_{j\in\mathbb{B}} \mathbf{Pr}\big\{r(0) = 0 \big| P(0|-1) = P_0, \Xi(-1) = i, \Xi(0) = j\big\} \\
&\qquad \times \mathbf{Pr}\big\{\Xi(0) = j \,\big|\, P(0|-1) = P_0, \Xi(-1) = i\big\} \\
&= \sum_{j\in\mathbb{B}} \mathbf{Pr}\big\{r(0) = 0 \big| \Xi(0) = j\big\} \mathbf{Pr}\big\{\Xi(0) = j \,\big|\, \Xi(-1) = i\big\} \\
&= \sum_{j\in\mathbb{B}} (1 - \phi_r(j)) \mathbf{Pr}\big\{\Xi(0) = j \,\big|\, \Xi(-1) = i\big\} = \nu(i). \quad (32)
\end{aligned}$$

The result follows upon replacing (30)–(32) into (29). □

Proof. [Theorem 6] We will use a stochastic Lyapunov function approach with candidate function V_k introduced in (27). By Lemma 10, we have

$$\begin{aligned}
0 &\le \mathbf{E}\big\{V_1 \,|\, Z(0) = (P_0, i)\big\} \\
&\le W + \big(\|A\|^2 \operatorname{tr} P_0 + \operatorname{tr} Q\big) \max_{i\in\mathbb{B}} \nu(i) \\
&= \|A\|^2 V_0 \max_{i\in\mathbb{B}} \nu(i) + \bar{\beta} \le \rho V_0 + \bar{\beta},
\end{aligned} \quad (33)$$

where $\rho \in [0, 1)$ is as in (22) and

$$\bar{\beta} \triangleq W + \operatorname{tr} Q \max_{i\in\mathbb{B}} \nu(i) \in [0, \infty).$$

Since (33) holds for all $Z(0)$ and, by Lemma 9, $\{Z\}_{\mathbb{N}_0}$ is Markovian, we can combine Theorem 2 in (Kushner, 1971, Ch. 8.4.2) with Theorem 2 in Tarn and Rasis (1976), to conclude that (33) is a sufficient condition for

$$0 \le \mathbf{E}\big\{V_k \,|\, Z(0) = Z\big\} \le \rho^k V_0 + \bar{\beta} \sum_{i=0}^{k-1} \rho^i, \quad \forall k \in \mathbb{N}. \quad (34)$$

On the other hand, since $P(k|k-1) \succeq 0$, it holds that $V_k \ge \|P(k|k-1)\|$, for all $k \in \mathbb{N}$. Therefore, upon noting that $P(0|-1)$ is given, (34) provides

$$\begin{aligned}
\mathbf{E}\big\{\|P(k|k-1)\|\big\} &\le \rho^k V_0 + \bar{\beta} \sum_{i=0}^{k-1} \rho^i = \rho^k V_0 + \bar{\beta} \frac{1-\rho^k}{1-\rho} \\
&\le (\rho V_0) \rho^{k-1} + \frac{\bar{\beta}}{1-\rho}, \quad \forall k \in \mathbb{N}.
\end{aligned}$$

Consequently, expression (19) holds with $\alpha = \rho V_0$ and $\beta = \bar{\beta}/(1-\rho)$. This proves the theorem. □

8. CONCLUSIONS

In this work we have studied stability properties of a Kalman filter when used for state estimation over a wireless sensor network. Since the radio links between the nodes are fading, even if alleviated by power control, packet drops may occur. We have established sufficient conditions, for the Kalman filter covariance matrix to be exponentially bounded in norm when the underlying network state is Markovian. Under this assumption, channel gains will be correlated over time, which is a suitable model when considering shadow fading.

In particular cases, the sufficient condition obtained reduces to stability results previously documented in the literature. Future work includes the development of power control and rerouting strategies for state estimation with sensor networks.

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