ON THE ROBUSTNESS OF PERIODIC SOLUTIONS IN RELAY FEEDBACK SYSTEMS

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Abstract: Structural robustness of limit cycles in relay feedback systems is studied. Motivated by a recent discovery of a novel class of bifurcations in these systems, it is illustrated through numerical simulation that small relay perturbations may change the appearance of closed orbits dramatically. It is shown analytically that certain stable periodic solutions in relay feedback systems are robust to relay perturbations.

Keywords: Limit cycles; Sliding Orbits; Perturbation analysis

1. INTRODUCTION

Relay feedback systems and, in general, nonsmooth feedback systems tend to self-oscillate (Tsypkin, 1984). Namely, the system evolution tends asymptotically towards stable periodic orbits or limit cycles. Recently, it has been shown that such solutions can undergo abrupt transitions when the system parameters are varied. This led to the discovery of an entirely novel class of bifurcations, involving the interaction between periodic solutions of the system and its discontinuity sets. Despite their widespread use in applications (Flügge–Lotz, 1953; Andronov *et al.*, 1965; Tsypkin, 1984; Åström and Hägglund, 1995; Norsworthy *et al.*, 1997), there are few analytical tools to characterize oscillations in relay feedback systems. For example, methods to assess their existence and stability properties are still the subject of much ongoing research (Åström, 1995; Megretski, 1996; Johansson *et al.*, 1997; Johansson *et al.*, 1999; di Bernardo *et al.*, 2000; Georgiou and Smith, 2000; Varigonda and Georgiou, 2001; Gonçalves *et al.*, 2001).

An interesting issue for the considered class of nonsmooth dynamical systems is the robustness properties of the solutions. Due to the discontinuous vector field, classical continuity results for smooth systems are not applicable. Still, it is important in applications to understand if a given solution is robust to unmodeled dynamics, external perturbations, and noise. While there are many results dealing with the robustness of smooth dynamical systems (e.g.,(Wiggins, 1990; Murdock, 1991; Kokotović *et al.*, 1999)), few papers seem to address this issue in the case of systems with nonsmooth vector fields. In the case of relay feedback systems, the available results deal with a quite

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restrictive class of systems where the transfer function is either close to an integrator (Georgiou and Smith, 2000) or to a second-order nonminimum phase system (Megretski, 1996). Singular perturbations for the smooth part of the system have also been studied (Fridman and Levant, 1996).

In this paper we are interested in the robustness of periodic solutions in relay feedback systems. In particular, we study the case when a system with an ideal relay exhibits an asymptotically stable periodic solution. Then we ask the question if a system with an imperfect implementation of the relay (modeled by a parameter $\varepsilon > 0$) will also have an asymptotically stable periodic solution. The considered relay implementations include relay with hysteresis, with finite gain (saturation), and with delayed switching. The problem is not trivial, especially, due to the nonsmooth characteristic of the relay. As an illustration, consider the approach, often suggested in the literature, of analyzing relay systems by approximating the relay by a continuous function. There are subtleties when taking the limit as the function tends to the characteristics of the relay. It was recently shown (Johansson *et al.*, 1999) that erroneous results have been derived in the literature when this limit is not dealt with properly.

The paper is outlined as follows. Relay feedback systems and the perturbations studied in the paper are introduced in Section 2. A motivating example is discussed in Section 3, where it is shown that several interesting bifurcation scenarios appear due to sudden loss of structural stability. Section 4 presents results on perturbations of relay feedback systems. It is shown that if a nominal system exhibits a stable periodic solution, then so will an ε-perturbed system under certain structures of the perturbation. Some concluding remarks and a discussion on future work are presented in Section 5.

2. RELAY FEEDBACK SYSTEMS

Consider a nominal relay feedback system

$$\Sigma_0 : \begin{cases} \dot{x} = Ax + Bu \\ y = Cx \\ u = -\operatorname{sgn} y, \end{cases}$$

where (A,B,C) defines a SISO linear time-invariant system of order $n \geq 1$. The relay, defined by the sign function, allows for sliding modes by the set-valued assignment $\operatorname{sgn} 0 \in [-1,1]$ and the interpretation of solutions (trajectories) in the sense of Filippov (Filippov, 1988). A solution $x : [0,\infty) \to \mathbb{R}^n$ of Σ_0 is periodic if there exists a (smallest) period time $T > 0$ such that $x(t+T) = x(t)$ for all $t > 0$. It is called symmetric if $x(t+T/2) = -x(t)$ for all $t \geq 0$. The switching plane is defined as

$$S = \{x \in \mathbb{R}^n : Cx = 0\}.$$

A periodic solution x is called simple if the closed orbit $\mathscr{L} = \{z \in \mathbb{R}^n : \exists t \geq 0, z = x(t)\}$ (i) intersects S only twice and (ii) is transversal to S at the intersection points. Note that the condition on transversal intersections is not fulfilled for so called sliding orbits (di Bernardo *et al.*, 2000). The following result gives conditions for existence and stability of periodic solutions (Åström, 1995; Varigonda and Georgiou, 2001). Note that stability refers to exponential stability throughout the paper.

Lemma 2.1. The system Σ_0 has a simple symmetric periodic solution with half-period t^* if and only if

$$f(t) > 0, \qquad 0 < t < t^*$$
$$f(t^*) = 0, \quad \frac{df}{dt}(0) > 0, \quad \frac{df}{dt}(t^*) < 0,$$

where

$$f(t) = Ce^{At}x^* - CA^{-1}(e^{At} - I)B$$
$$x^* = (e^{At^*} + I)^{-1}A^{-1}(e^{At^*} - I)B.$$

Moreover, it is stable if all eigenvalues of the Jacobian

$$W = \left(I - \frac{wC}{Cw}\right)e^{At^*}, \qquad w = (e^{At^*} + I)^{-1}e^{At^*}B$$

are in the open unit disc.

Note that the point x^* is the intersection point with the switching plane. Extensions of the result are discussed in (Johansson *et al.*, 1997; Johansson *et al.*, 1999; di Bernardo *et al.*, 2000; Varigonda and Georgiou, 2001).

Next we introduce the three alternative relay perturbations that we study in the paper. A relay feedback system with hysteresis $\varepsilon > 0$ is denoted

$$\Sigma_\varepsilon^H : \begin{cases} \dot{x} = Ax + Bu \\ y = Cx \\ u = -\operatorname{sgn}_\varepsilon^H y, \end{cases}$$

where the relay is defined as

$$u(t) = -\operatorname{sgn}_\varepsilon^H y(t)$$
$$= \begin{cases} -1, & y(t) > \varepsilon \text{ or } \big(-\varepsilon < y(t) < \varepsilon,\, u(t-) = -1\big) \\ 1, & y(t) < -\varepsilon \text{ or } \big(-\varepsilon < y(t) < \varepsilon,\, u(t-) = 1\big). \end{cases}$$

A relay feedback system with the relay replaced by a saturation with steep slope $1/\varepsilon > 0$ is given by

$$\Sigma_\varepsilon^S : \begin{cases} \dot{x} = Ax + Bu \\ y = Cx \\ u = -\operatorname{sgn}_\varepsilon^S y, \end{cases}$$

where the relay is defined as

$$u(t) = -\operatorname{sgn}_\varepsilon^S y(t) = \begin{cases} -1, & \text{if } y(t) > \varepsilon \\ -y(t)/\varepsilon, & \text{if } -\varepsilon < y(t) < \varepsilon \\ 1, & \text{if } y(t) < -\varepsilon. \end{cases}$$

A relay feedback system with switching delayed $\varepsilon > 0$ amount of time is defined as

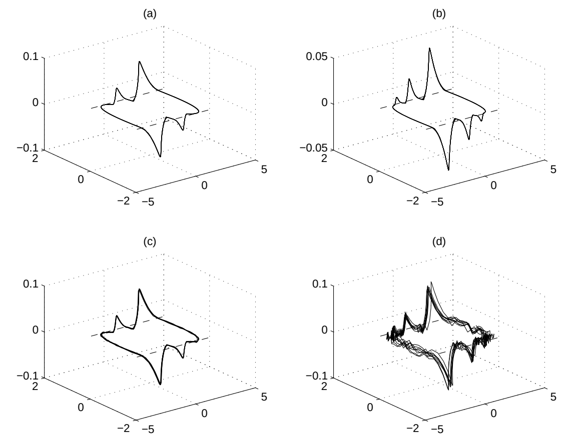

Fig. 1. Oscillations of the third-order relay systems Σ_0 (a and b) and Σ_ε^H (c and d). The parameter values are $\zeta = 0.05, \lambda = \rho = -\sigma = 1$ and **(a)** $\omega = 10.3$, $\varepsilon = 0$, **(b)** $\omega = 12$, $\varepsilon = 0$, **(c)** $\omega = 10.3$, $\varepsilon = 1/1000$, **(d)** $\omega = 10.3$, $\varepsilon = 1/100$.

$$\Sigma_\varepsilon^D : \begin{cases} \dot{x} = Ax + Bu \\ y = Cx \\ u = -\operatorname{sgn}_\varepsilon^D y, \end{cases}$$

where the relay is simply

$$u(t) = -\operatorname{sgn}_\varepsilon^D y(t) = -\operatorname{sgn} y(t-\varepsilon).$$

It should be noticed that the definitions for periodic solutions for Σ_0 directly generalize to the perturbed systems Σ_ε^D, Σ_ε^H, and Σ_ε^S.

3. MOTIVATING EXAMPLES

A third-order relay feedback system recently studied in (di Bernardo *et al.*, 2000; Kowalczyk and di Bernardo, 2001*a*; Kowalczyk and di Bernardo, 2001*b*) is now used as a representative example. The linear dynamics is given by

$$A = \begin{pmatrix} -(2\zeta\omega+\lambda) & 1 & 0 \\ -(2\zeta\omega\lambda+\omega^2) & 0 & 1 \\ -\lambda\omega^2 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 \\ 2\sigma\rho \\ \rho^2 \end{pmatrix}$$
$$C = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix},$$

which corresponds to the transfer function

$$C(sI-A)^{-1}B = \frac{s^2+2\sigma\rho s+\rho^2}{(s^2+2\zeta\omega s+\omega^2)(s+\lambda)}.$$

This system has been shown to undergo several bifurcation phenomena, which can lead to the occurrence of deterministic chaos (see (Kowalczyk and di Bernardo, 2001*b*) for a complete description of the bifurcation diagram). This would seem to indicate that periodic solutions of relay systems are sensitive to parameter variations and external disturbances.

In the simplest case, a change in the topology of the solution of Σ_0 can be observed as the parameters are varied. An example is shown in Figs. 1(a)–(b), where the transition is depicted from a periodic solution characterized by two segments of sliding motion each half-period to one containing three sections of sliding. More complex scenarios are also possible corresponding to a sudden loss of structural stability. The system can for example exhibit so-called period-doubling cascades to chaos (Kowalczyk and di Bernardo, 2001*b*) or in some cases an abrupt transition from regular to chaotic motion (Verghese and Banerjee, 2001). The occurrence of these phenomena has been recently explained in the literature as due to the occurrence of new bifurcations, unique to nonsmooth systems. The formation of periodic solutions with sliding (or *sliding orbits*), for example, has been explained by identifying so-called sliding bifurcations (di Bernardo *et al.*, 2000). These are due to interactions between periodic orbits of the system and regions on the discontinuity set where sliding is possible. The existence of unexpected transitions involving self-oscillations of relay feedback systems motivates the study of how persistent periodic solutions are. We restrict our attention to the effects of perturbations to the relay characteristics.

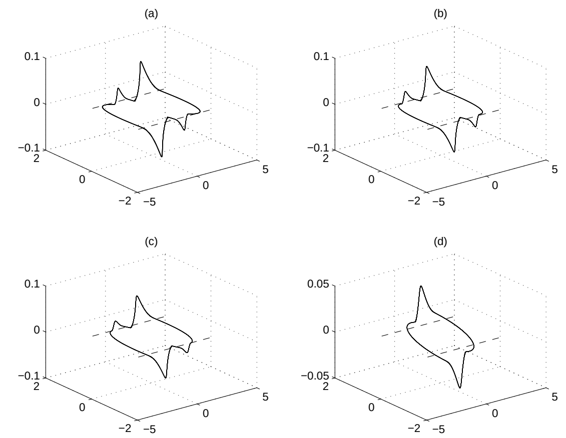

Fig. 2. Oscillations of perturbed third-order relay system Σ_ε^S with the same parameters as in Fig. 1. The perturbation is **(a)** $\varepsilon = 0$, **(b)** $\varepsilon = 1/500$, **(c)** $\varepsilon = 1/250$, **(d)** $\varepsilon = 1/100$.

Our numerics seem to suggest that oscillations in relay feedback systems are unexpectedly robust to perturbations of the relay characteristic. Fig. 1(c) shows, for instance, that the orbit characterized by two sections of sliding motion for Σ_0 depicted in Fig. 1(a) is robust to a small hysteresis (Σ_ε^H with $\varepsilon = 1/1000$). We see, though, that as the perturbation is increased the effects of the hysteresis cannot be neglected (Fig. 1(d)). Nevertheless, the influence of the underlying unperturbed orbit remains clearly visible.

Similar effects as in Fig. 1 are shown in Fig. 2 but for Σ_ε^S, in which case the system is perturbed by substituting the relay element with a finite gain saturation. Again we see that for relatively small value of the perturbation (high value of the gain), the perturbed orbits (Fig. 2(b)–(c)) stay close to the unperturbed one (Fig. 2(a)). Lower values of the gain though cause the transition to the different orbit depicted in Fig. 2(d).

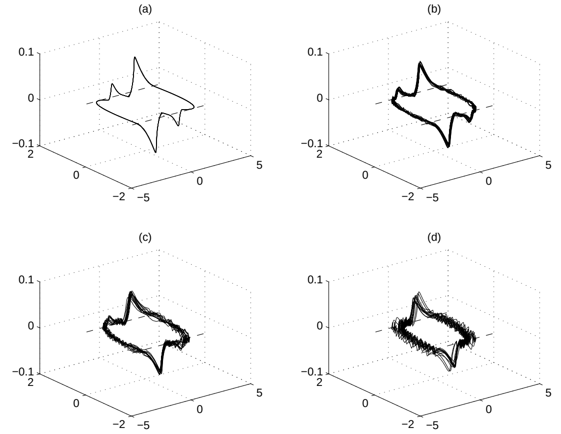

Fig. 3. Oscillations of the delayed relay system Σ_ε^D for the same parameter values as in Fig. 1. **(a)** $\varepsilon = 0$, **(b)** $\varepsilon = 1/200$, **(c)** $\varepsilon = 1/125$, **(d)** $\varepsilon = 1/100$.

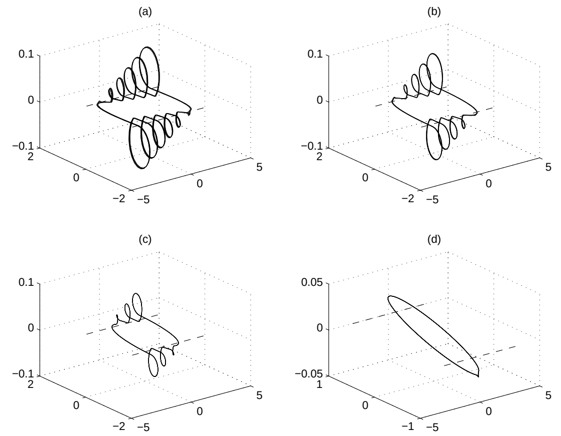

Fig. 4. Oscillations of perturbed third-order relay system Σ_ε^S with parameters $\zeta = -0.07, \lambda = 0.05, \rho = -\sigma = 1$ and $\omega = 10$. **(a)** $\varepsilon = 0$, **(b)** $\varepsilon = 1/200$, **(c)** $\varepsilon = 1/100$, **(d)** $\varepsilon = 1/50$.

Note that the persistence observed in the system is quite remarkable. Substituting the relay with a saturation prevents the occurrence of sliding mode without causing a destruction of the unperturbed solution structure.

This structural robustness is also observed in the case of Σ_ε^D, where the relay is perturbed by adding a small delay. Fig. 3 shows how the periodic orbit under investigation varies as the delay is increased. Despite the onset of high-frequency oscillations, the structure of the unperturbed orbit is still preserved.

More dramatic effects are observed when the robustness of a more complex dynamical behavior is investigated. When the chaotic attractor shown in Fig. 4(a) is perturbed by substitution of the relay with a high gain saturation, its topology changes to the one shown in Fig. 4(b) (characterized by a lower number of lobes). Further variation of the gain, causes a further reduction of the lobes (Fig. 4(c)) followed by the appearance of a stable *asymmetric* periodic solution (Fig. 4(d)).

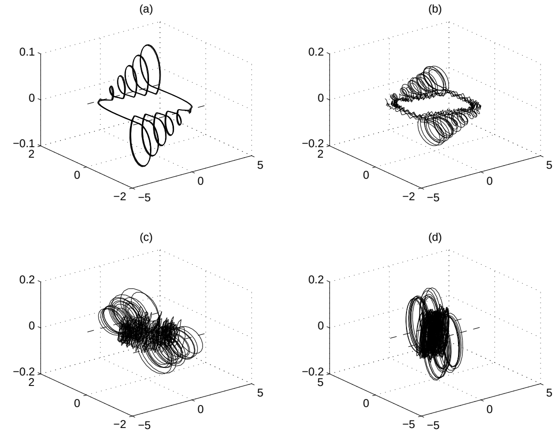

Fig. 5. Oscillations of the hysteresis perturbed relay system Σ_ε^H for the same parameter values as in Fig. 4, but with **(a)** $\varepsilon = 0$, **(b)** $\varepsilon = 1/100$, **(c)** $\varepsilon = 1/20$, **(d)** $\varepsilon = 1/10$.

The effects of a small hysteresis on the same chaotic attractor are even more evident as shown in Fig. 5, where Σ_ε^H. Here we see the attractor structure changing rapidly as the perturbation is increased.

The simulations reported above highlight the need for appropriate theoretical tools to systematically carry out the robustness analysis of oscillations in relay systems. In what follows, perturbation analysis of so-called simple periodic solutions is discussed. These intersect the switching plane transversally, which make them easier to analyze using classical Poincaré techniques. Note that it seems that tangential intersections plays an important role in some of the bifurcation phenomena illustrated above, cf., bifurcation analysis in (di Bernardo *et al.*, 2000). The robustness analysis of periodic solution that hits or leaves the switching plan tangentially will be studied in future work.

4. PERTURBATION ANALYSIS

In this section we study different perturbations of the nominal relay feedback system Σ_0. Given some rather non-restrictive assumptions, we will see that a stable periodic solution of Σ_0 is persistent, in the sense that the perturbed system Σ_ε^P also has a stable periodic solution regardless of the perturbation P. The following theorem summarizes the result of the section.

Theorem 4.1. Suppose the relay feedback system Σ_0 has a simple symmetric periodic solution with a strictly stable Jacobian (as defined in Lemma 2.1). Then, there exists $\varepsilon_0 > 0$ such that for each $\varepsilon \in (0, \varepsilon_0)$ the perturbed relay feedback systems Σ_ε^H, Σ_ε^S, and Σ_ε^D all have simple symmetric stable periodic solutions.

The proof is rather straightforward and follows from the lemmas below, where each relay perturbation is treated separately. The proof is based on techniques used in the recent literature on relay feedback systems, e.g., (Åström, 1995). Throughout the section, we make the following standing assumption.

Assumption 4.1. The relay feedback system Σ_0 has a simple symmetric periodic solution with half-period t^*. Moreover, all eigenvalues of W (defined in Lemma 2.1) are inside the unit disk.

Consider the relay feedback system with hysteresis Σ_ε^H. We note that with a straightforward modification of Lemma 2.1 the following result holds, cf., (Åström, 1995).

Lemma 4.1. The system Σ_ε^H has a stable simple symmetric periodic solution with half-period τ if (i)

$$f^H(t,\varepsilon) > 0, \qquad 0 < t < \tau$$
$$f^H(\tau,\varepsilon) = 0, \quad \frac{df^H}{dt}(0,\varepsilon) > 0, \quad \frac{df^H}{dt}(\tau,\varepsilon) < 0$$

where

$$f^H(t,\varepsilon) = Ce^{At}z^* - CA^{-1}(e^{At} - I)B + \varepsilon$$
$$z^* = (e^{A\tau} + I)^{-1}A^{-1}(e^{A\tau} - I)B,$$

and (ii) all eigenvalues of the Jacobian

$$W^H(\varepsilon) = \left(I - \frac{w^H C}{Cw^H}\right)e^{A\tau}$$
$$w^H(\varepsilon) = (e^{A\tau} + I)^{-1}e^{A\tau}B$$

are in the open unit disc.

If the nominal system Σ_0 generates a closed orbit as specified in Lemma 2.1, one may ask if also Σ_ε^H generates one. Next we prove that this is the case if $\varepsilon > 0$ is small.

Lemma 4.2. There exists $\varepsilon_0 > 0$ such that for each $\varepsilon \in (0,\varepsilon_0)$ the system Σ_ε^H has a simple symmetric stable periodic solution.

Proof: By Assumption 4.1 and Lemma 2.1, we have that $f(t^*) = 0$, or, equivalently, that $f^H(t^*,0) = 0$. Since

$$\frac{\partial f^H}{\partial t}(t^*,0) = \frac{\partial f}{\partial t}(t^*) \neq 0$$

from the assumption on transversal intersections, it follows from the Implicit Function Theorem that there exists $\varepsilon_1 > 0$ and a unique $\tau(\varepsilon)$ such that $f^H(\tau(\varepsilon),\varepsilon) = 0$ for all $\varepsilon \in (0,\varepsilon_1)$. The assumption on transversal intersections also leads to that the inequality assumptions in Lemma 4.1 hold. Stability is guaranteed from that W^H in Lemma 4.1 is identical to the matrix W in Lemma 2.1, but with t^* replaced by τ. Hence, by continuity and the assumption that the eigenvalues of W are strictly inside the unit disk, there thus exists $\varepsilon_0 \in (0,\varepsilon_1)$ such that the result holds. ■

Consider the perturbed relay feedback system Σ_ε^S, where the relay is replaced by a saturation with steep slope. Introduce the notation ϕ_+ for the flow of $\dot{x} = Ax + B$, ϕ_- for the flow of $\dot{x} = Ax - B$, and ϕ_ε for the flow of $\dot{x} = (A - BC/\varepsilon)x$. The following result similar to Lemma 2.1 then holds.

Lemma 4.3. The system Σ_ε^S has a stable simple symmetric periodic solution with half-period $\tau_1 + \tau_2 + \tau_3$ if (i)

$$\begin{aligned} &0 < f_1^S(t,\varepsilon) < \varepsilon, && 0 < t < \tau_1 \\ &\varepsilon < f_2^S(t,\varepsilon), && 0 < t < \tau_2 \\ &0 < f_3^S(t,\varepsilon) < \varepsilon, && 0 < t < \tau_3 \end{aligned}$$

$$f_1^S(\tau_1,\varepsilon) = \varepsilon, \quad f_2^S(\tau_2,\varepsilon) = \varepsilon, \quad f_3^S(\tau_3,\varepsilon) = 0$$

$$\begin{aligned} &\frac{df_1^S}{dt}(0,\varepsilon) > 0, && \frac{df_1^S}{dt}(\tau_1,\varepsilon) > 0 \\ &\frac{df_2^S}{dt}(0,\varepsilon) > 0, && \frac{df_2^S}{dt}(\tau_2,\varepsilon) < 0 \\ &\frac{df_3^S}{dt}(0,\varepsilon) < 0, && \frac{df_3^S}{dt}(\tau_3,\varepsilon) < 0 \end{aligned}$$

where

$$f_1^S(t,\varepsilon) = C\phi_\varepsilon(t,z^*), \; f_2^S(t,\varepsilon) = C\phi_-(t,z_1^*)$$
$$f_3^S(t,\varepsilon) = C\phi_\varepsilon(t,z_2^*)$$
$$z^* = -\phi_\varepsilon(\tau_3,z_2^*), \; z_1^* = \phi_\varepsilon(\tau_1,z^*)$$
$$z_2^* = \phi_-(\tau_2,z_1^*),$$

and (ii) all eigenvalues of the Jacobian

$$W^S(\varepsilon) = W_3^S(\varepsilon)W_2^S(\varepsilon)W_1^S(\varepsilon)$$

are in the open unit disc with

$$W_1^S(\varepsilon) = \left(I - \frac{M_1 Az^* C}{CM_1 Az^*}\right)M_1$$
$$W_2^S(\varepsilon) = \left(I - \frac{w^S C}{Cw^S}\right)e^{A\tau_2}$$
$$W_3^S(\varepsilon) = \left(I - \frac{M_3(Az_2^* - BC)C}{CM_3(Az_2^* - BC)}\right)M_3$$
$$M_1 = e^{(A-BC/\varepsilon)\tau_1}, \quad M_3 = e^{(A-BC/\varepsilon)\tau_3}$$
$$w^S(\varepsilon) = e^{A\tau_2}(Az_1^* - B).$$

Proof: Existence follows directly by the construction of a periodic solution, by considering a trajectory starting in a point $z^* \in S$, flowing to $z_1^* = \phi_\varepsilon(\tau_1,z^*)$, further to $z_2^* = \phi_-(\tau_2,z_1^*)$, and finally to $\phi_\varepsilon(\tau_3,z_2^*) = -z^*$. The existence of a closed orbit follows by symmetry. The Jacobian W^S of the corresponding Poincaré map is straightforward to derive. Stability of W^S implies stability of the periodic solution. The factors $W_1^S(\varepsilon)$ and $W_2^S(\varepsilon)$ are worth some discussion. Let us take $W_1^S(\varepsilon)$ for instance. It is the Jacobian of the Poincaré map $P_1 : S \to S_\varepsilon = \{x \in \mathbb{R}^n : Cx = \varepsilon\}$ defined by $P_1(x) = \phi_\varepsilon(\tau_1(x),x)$, where $\tau_1(x)$ is the transition time. In particular, $\tau_1(z^*) = \tau_1$. A standard derivation as in, e.g. (Åström, 1995), shows that

$$DP_1(z^*) = \left(I - \frac{M_1(\varepsilon)(A - BC/\varepsilon)z^* C}{CM_1(\varepsilon)(A - BC/\varepsilon)z^*}\right)M_1(\varepsilon)$$

Since, $z^* \in S$, we have $Cz^* = 0$ and hence $(A - BC/\varepsilon)z^* Az^*$. This gives the expression for $W_1^S(\varepsilon)$ in the statement of the lemma. Similarly, $P_3 : S_\varepsilon \to S$ defined by $P_3(x) = \phi_\varepsilon(\tau_3(x), x)$, where $\tau_3(z_2^*) = \tau_3$. We get

$$DP_3(z_2^*) = \left(I - \frac{M_3(\varepsilon)(A - BC/\varepsilon)z_2^* C}{CM_3(\varepsilon)(A - BC/\varepsilon)z_2^*} \right) M_3(\varepsilon)$$

Using that $z_2^* \in S_\varepsilon$ gives $(A - BC/\varepsilon)x_2^* = Az_2^* - BC$ and the expression for $W_3^S(\varepsilon)$ follows. ∎

The robustness result is now as follows.

Lemma 4.4. There exists $\varepsilon_0 > 0$ such that for each $\varepsilon \in (0, \varepsilon_0)$ the system Σ_ε^S has a simple symmetric stable periodic solution.

Proof: Consider $x^* \in S$ as given in Lemma 2.1. We will show that there exists a contraction mapping for Σ_ε^S that maps a neighborhood of x^* back to itself, when $\varepsilon > 0$ is small. The trajectory of Σ_ε^S passing through the fixed point z^* of that map will then be shown to fulfill the conditions in Lemma 4.3.

Since the nominal system Σ_0 has a simple periodic solution by assumption, intersection with S is transversal and thus there exists a neighborhood $U \subset S$ of x^* such that $C(Ax \pm B) > 0$ for all $x \in U$. Define a map $P : U \to U$ as $P(x) = -\phi_-(\tau(x), x)$, where $\tau(x)$ is the time to first intersection with S for the trajectory of Σ_0 passing through the point $x \in U$.

Consider a point $x_0 = x^* + \delta \in U$ under the flow of Σ_ε^S. For small $t > 0$,

$$\begin{aligned}\phi_\varepsilon(t, x_0) &= x_0 + (A - BC/\varepsilon)x_0 t + \mathcal{O}(t^2) \\ &= x_0 + Ax_0 t + \mathcal{O}(t^2),\end{aligned}$$

since $Cx_0 = 0$. The time $\tau_1 > 0$ at which the trajectory for the first time intersects $S_\varepsilon = \{x \in \mathbb{R}^n : Cx = \varepsilon\}$ is for small $\varepsilon > 0$ thus given by

$$\tau_1 = \frac{\varepsilon}{CAx_0} + \mathcal{O}(\varepsilon^2).$$

Note that $CAx_0 > 0$, since $C(Ax_0 + B) > 0$ and $C(Ax_0 - B) > 0$ due to the assumption of transversal flow in U. This shows that $x_1 = \phi_\varepsilon(\tau_1(x_0), x_0) = x_0 + \mathcal{O}(\varepsilon)$. If x_0 is close to x^* then the dynamics of Σ_ε^S maps $x_1 \mapsto x_2 = \phi_-(\tau_2(x_1), x_1) \in S_\varepsilon$ continuously and the vector field intersects S_ε transversally at x_2. A similar argument as above shows that $x_3 = \phi_\varepsilon(\tau_3(x_2), x_2) = x_2 + \mathcal{O}(\varepsilon)$. It follows that there exists a neighborhood $V \subset U$ of x^* such that the map $P_\varepsilon : V \to S$ given by $P_\varepsilon(x_0) = -x_3$, with x_3 specified above, is well defined, smooth, and maps x^* to $x^* + \mathcal{O}(\varepsilon)$. It is easy to show that the Jacobian of the map $x_0 \mapsto x_1 = \phi_\varepsilon(\tau_1(x_0), x_0)$ is given by

$$\tilde{W}_1^S(\varepsilon; x_0) = \left(I - \frac{\tilde{M}_1 Ax_0 C}{CA\tilde{M}_1 x_0} \right) \tilde{M}_1.$$

where $\tilde{M}_1 = e^{(A - BC/\varepsilon)\tau_1(x_0)}$. The corresponding maps $x_1 \mapsto x_2 = \phi_-(\tau_2(x_1), x_1)$ and $x_2 \mapsto x_3 = \phi_\varepsilon(\tau_3(x_2), x_2)$ have Jacobians $\tilde{W}_2^S(\varepsilon; x_1)$ and $\tilde{W}_3^S(\varepsilon; x_2)$, with expressions similar to W_2^S and W_3^S in Lemma 4.3. Note that all Jacobians are well defined for sufficiently small ε.

Let $B_r(x^*) \subset U$ be a ball with radius $r > 0$ and center in x^*. For $x_0 = x^* + \delta \in B_r(x^*)$, we have by series expansion

$$P_\varepsilon(x^* + \delta) = x^* + \mathcal{O}(\varepsilon) + \tilde{W}^S(\varepsilon; x^*)\delta + \mathcal{O}(\delta^2), \quad (1)$$

where $\tilde{W}^S(\varepsilon; x^*) = -\tilde{W}_3^S(\varepsilon; x_2^*)\tilde{W}_2^S(\varepsilon; x_1^*)\tilde{W}_1^S(\varepsilon; x^*)$ with x_1^* and x_2^* being the corresponding intersection points with S_ε. Note that for small $\varepsilon > 0$, both $\tilde{W}_1^S(\varepsilon; x^*)$ is approximately equal to the identity map on S (up to $\mathcal{O}(\varepsilon)$). This follows since $\tilde{M}_1 = e^{(A - BC/\varepsilon)\tau_1(x_0)} = e^{-BC/(CAx_0)} + \mathcal{O}(\varepsilon)$, and thus for $\delta \in S$, we have

$$\begin{aligned}\tilde{W}_1^S(\varepsilon; x^*)\delta &= \left(I - \frac{\tilde{M}_1 Ax_0 C}{CA\tilde{M}_1 x_0} \right) (e^{-BC/(CAx_0)} + \mathcal{O}(\varepsilon))\delta \\ &= \left(I - \frac{\tilde{M}_1 Ax_0 C}{CA\tilde{M}_1 x_0} \right) (\delta + \mathcal{O}(\varepsilon)) = \delta + \mathcal{O}(\varepsilon).\end{aligned}$$

We can in a similar way show that $\tilde{W}_1^S(\varepsilon; x_2^*)$ is approximately equal to the identity map on S and that $\tilde{W}_2^S(\varepsilon; x_1^*) = W + \mathcal{O}(\varepsilon)$. Hence, $\tilde{W}^S(\varepsilon; x^*)$ is stable for small enough ε, so that Equation (1) gives $P_\varepsilon\big(B_r(x^*)\big) \subset B_r(x^*)$ if $\varepsilon \ll r \ll 1$.

Next we show that P_ε has a unique fixed point $z^* \in B_r(x^*)$ if r is sufficiently small. Consider two points $x, x + \delta \in B_r(x^*)$, and note that

$$P_\varepsilon(x + \delta) = P_\varepsilon(x) + \tilde{W}^S(\varepsilon; x)\delta + \mathcal{O}(\delta^2).$$

Since $\|\delta\|$ is small and $\tilde{W}^S(\varepsilon; x)$ is stable for all $x \in B_r(x^*)$, it holds that $\|\tilde{W}^S(\varepsilon; x)\delta + \mathcal{O}(\delta^2)\| < \|\delta\|$. Hence, P_ε is a contraction on $B_r(x^*)$ if r is sufficiently small. It thus follows that P_ε has a unique fixed point $z^* \in B_r(x^*)$.

It remains to show that the trajectory of Σ_ε^S passing through z^* fulfills the conditions of Lemma 4.3, and hence generates a simple symmetric stable periodic solution. The stability follows from that $W^S(\varepsilon; z^*)$ is stable as we argued above. The other conditions follows by a straightforward continuity argument, since the flows ϕ_ε and ϕ_- are smooth and Σ_0 has a simple symmetric stable periodic solution. ∎

Consider the perturbed relay feedback system Σ_ε^D, where the switching is delayed a short amount of time.

Lemma 4.5. The system Σ_ε^D has a stable simple symmetric periodic solution with half-period $\tau + \varepsilon$ if (i)

$$\begin{aligned}&f_1^D(t, \varepsilon) > 0, \qquad 0 < t < \varepsilon \\ &f_2^D(t, \varepsilon) > 0, \qquad 0 < t < \tau \\ &f_2^D(\tau, \varepsilon) = 0, \quad \frac{df_1^D}{dt}(0, \varepsilon) > 0, \quad \frac{df_2^D}{dt}(\tau, \varepsilon) < 0,\end{aligned}$$

where

$$\begin{aligned}f_1^D(t, \varepsilon) &= C\phi_+(t, z^*) \\ f_2^D(t, \varepsilon) &= C\phi_-(t, \phi_+(\varepsilon, z^*)) \\ z^* &= (e^{A(\tau+\varepsilon)} + I)^{-1} A^{-1} (2e^{A\tau} - e^{A(\tau+\varepsilon)} - I)B,\end{aligned}$$

and (ii) all eigenvalues of the Jacobian

$$W^D(\varepsilon) = W_2^D(\varepsilon) W_1^D(\varepsilon)$$

are in the open unit disc with

$$W_1^D(\varepsilon) = e^{A\varepsilon}, \quad W_2^D(\varepsilon) = \left(I - \frac{w^D C}{C w^D}\right) e^{A\tau}$$
$$w^D(\varepsilon) = e^{A\tau}(A\phi_+(\varepsilon, z^*) - B).$$

Proof: Existence follows by the construction of a periodic solution starting in z^*. Stability follows from explicitly deriving W^D, which is the Jacobian of the corresponding Poincaré map. ∎

The robustness result is now as follows.

Lemma 4.6. There exists $\varepsilon_0 > 0$ such that for each $\varepsilon \in (0, \varepsilon_0)$ the system Σ_ε^D has a simple symmetric stable periodic solution.

Proof: Similar to the proof Lemma 4.4, but with application of Lemma 4.5. Note that the Jacobian $W^D(\varepsilon) = W_2^D(\varepsilon) W_1^D(\varepsilon)$ tends to W as ε tends to zero. ∎

5. CONCLUSIONS AND FUTURE WORK

Perturbation analysis in relay feedback systems was discussed. It was shown that stable simple symmetric periodic solutions are persistent under small variations in the relay characteristic. Simulations showed that if the orbits are not simple (i.e., do not intersect the switching plane transversally twice per period), then sensitive solutions may appear. Examples of this include so-called sliding orbits. Future work include studying perturbations of sliding orbits in detail. The analysis in the paper was straightforward and directly extends techniques developed in (Åström, 1995; Johansson *et al.*, 1999; Varigonda and Georgiou, 2001; di Bernardo *et al.*, 2000). The proof technique can be extended to more general piecewise affine systems. It is interesting to consider relay feedback systems with other imperfections, such as model errors in the linear dynamics and unmodeled dynamics. Note that it is straightforward to extend Theorem 4.1 to a class of system Σ_ε^M, which has an ideal relay but the linear system is replaced by smooth functions $A(\varepsilon), B(\varepsilon), C(\varepsilon)$, such that $(A(0), B(0), C(0)) = (A, B, C)$.

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