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Probabilistic Characterization of Target Set and Region of Attraction for Discrete-time Control Systems

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Abstract— **This paper proposes a new notion of stabilization in probability for discrete-time stochastic systems that may be with unbounded disturbances and bounded control input. This new notion builds on two sets: target set and region of attraction. The target set is a set within which the controller is able to keep the system state with a certain probability. The region of attraction is a set from which the controller is able to drive the system state to the target set with a prescribed probability. We investigate the probabilistic characterizations of these two sets for linear stochastic control systems. We provide sufficient conditions for a compact set to be a target set with a given horizon and probability level. Given a target set, we use two methods to characterize the region of attraction: one is based on the solution to a stochastic optimal first-entry time problem while the other is based on stochastic backward reachable sets. For linear scalar systems, we provide analytic representations for the target set and the region of attraction. Simulations are given to illustrate the effectiveness of the theoretical results.**

I. Introduction

Stabilization of a dynamical controlled system is a fundamental problem in systems and control. This problem addresses how to design the control input such that the state of the dynamical system is stable in some sense. In the past decades, there have been many research efforts on this subject [1]. For example, the stabilization of deterministic nonlinear systems is investigated in [2] and the robust stabilization of uncertain linear systems is studied in [3].

Stochastic modeling of dynamical systems is very important in many applications, such as engineering, biology, and economics. Stabilization of stochastic control systems has been widely studied by many literatures [4], [5]. Stabilization in probability is a well-known concept closely related to the concept of stability in probability. For the interested reader, please refer to [6] for a more detailed statement of stochastic stabilization and stability.

In [7], [8], the authors propose a new definition of stabilization in probability for continuous-time nonlinear stochastic systems. This definition consists of region of attraction, target set, and two probability levels. One probability level captures the minimal probability that the state remains in the target set under some admissible control input after entering it. The other one provides the minimal probability that the state is driven to enter the target set from the region of attraction under some admissible control input within a finite time. This new definition of stabilization in probability generalizes the regional stabilization of deterministic dynamical systems and stands as an intermediate notion between local and global stabilizations in probability.

In this paper, we extend the notion of stabilization in probability in [7], [8] for discrete-time stochastic systems, which also builds on the region of attraction and target set. We notice that the control input constraints exist in many applications, e.g., due to the actuator saturation, while the supports of many stochastic disturbances, e.g., Gaussian noise, are unbounded. This extension provides a way to establish the stabilization of systems with unbounded disturbances under a bounded control input, which is beyond the scope of the traditional stochastic stabilization. Different from [7], [8], the focus of this paper is on the characterizations of region of attraction and target set for linear stochastic control systems. The main contributions are summarized as follows.

- We show that the target set is consistent with a probabilistic controlled invariant set in our recent work [9], [10] and derive sufficient conditions for a compact set to be a target set with a given horizon and a probability level.
- We provide two methods to characterize the region of attraction. One is based on the solution of a stochastic optimal first-entry time problem, which gives an exact characterization of region of attraction. The other is based on stochastic backward reachable sets, which gives an overapproximation of region of attraction.
- For linear scalar systems, we provide analytic representations for the target set and the region of attraction.

The remainder of this paper is organized as follows. Section II provides the problem statement. Section III characterizes the target set while Section IV characterizes the region of attraction. An example in Section V illustrates the effectiveness of the results. Section VI concludes this paper.

Notation. Let $\mathbb{N}$ denote the set of nonnegative integers and $\mathbb{R}$ denote the set of real numbers. For some $q, s \in \mathbb{N}$ and $q < s$, let $\mathbb{N}_{\geq q}$ and $\mathbb{N}_{[q,s]}$ denote the sets $\{r \in \mathbb{N} \mid r \geq q\}$ and $\{r \in \mathbb{N} \mid q \leq r \leq s\}$, respectively. Let $\bar{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$. For two sets $\mathbb{X}$ and $\mathbb{Y}$, $\mathbb{X} \setminus \mathbb{Y} = \{x \mid x \in \mathbb{X}, x \notin \mathbb{Y}\}$. When $\leq, \geq$, $<$, and $>$ are applied to vectors, they are interpreted elementwise. $\Pr$ denotes the probability and E the expection. For a

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set $\mathbb{X}$, $\mathcal{B}(\mathbb{X})$ denotes the Boreal σ-algebra generated by $\mathbb{X}$. The indicator function of a set $\mathbb{X}$ is denoted by $\mathbf{1}_{\mathbb{X}}(x)$, that is, if $x \in \mathbb{X}$, $\mathbf{1}_{\mathbb{X}}(x) = 1$ and otherwise, $\mathbf{1}_{\mathbb{X}}(x) = 0$.

II. Problem Formulation

Consider a discrete-time linear control system

$$x_{k+1} = Ax_k + Bu_k + w_k, \tag{1}$$

where $x_k \in \mathbb{R}^{n_x}$ is the state, $u_k \in \mathbb{R}^{n_u}$ the control input, and $w_k \in \mathbb{R}^{n_x}$ the stochastic disturbance.

We further assume that w_k, $\forall k \in \mathbb{N}$, are zero-mean and independent and identically distributed random variables with density function $f(\cdot)$ and support $\mathbb{W}$. Let the state space be $\mathbb{X} = \mathbb{R}^{n_x}$ and the control input space be a compact set $\mathbb{U} \subset \mathbb{R}^{n_u}$, i.e., $u_k \in \mathbb{U}, \forall k \in \mathbb{N}$. Next, we define Markov policies for the system (1).

Definition 2.1: Given any $N \in \mathbb{N}$, a Markov policy $\boldsymbol{\mu}$ for the system (1) is a sequence $\boldsymbol{\mu} = (\mu_0, \mu_1, \ldots, \mu_{N-1})$ of universally measurable maps:

$$\mu_k : \mathbb{X} \to \mathbb{U}, \forall k \in \mathbb{N}_{[0,N-1]}.$$

When $N = \infty$, a Markov policy is a sequence of $\boldsymbol{\mu} = (\bar{\mu}, \bar{\mu}, \ldots)$ of universally measurable map $\bar{\mu} : \mathbb{X} \to \mathbb{U}$.

We remark that the Markov policy $\boldsymbol{\mu}$ for $N = \infty$ is called stationary policy in the literature. Let $\mathcal{M}$ be the set of Markov policies. Given $N \in \bar{\mathbb{N}}$, an initial condition x_0 and a Markov policy $\boldsymbol{\mu} \in \mathcal{M}$, one execution generates a sequence of states $(x_0, x_1, \ldots, x_N)$. In the following, with some abuse of notation, we also use $[0, N]$ for $N = \infty$.

Following the definition of stabilization in probability for continuous-time nonlinear stochastic systems in [7], [8], we define the stabilization in probability for the discrete-time system (1).

Definition 2.2: (Stabilization in Probability) Given $\alpha, \beta \in (0, 1]$, $N \in \bar{\mathbb{N}}$ and $\mathbb{Q}, \mathbb{P} \in \mathcal{B}(\mathbb{X})$ with $\mathbb{Q} \subseteq \mathbb{P}$, the system (1) is said to be $(\mathbb{P}, \mathbb{Q}, N, \alpha, \beta)$-stabilizable in probability if the following conditions hold:

$$\inf_{x_0 \in \mathbb{Q}} \sup_{\boldsymbol{\mu} \in \mathcal{M}} \Pr\{\forall k \in \mathbb{N}_{[0,N]}, x_k \in \mathbb{Q}\} \geq \alpha, \tag{2a}$$

$$\inf_{x_0 \in \mathbb{P}} \sup_{T \in \mathbb{N}} \sup_{\boldsymbol{\mu} \in \mathcal{M}} \Pr\{\exists k \in \mathbb{N}_{[0,T]}, x_k \in \mathbb{Q}\} \geq \beta. \tag{2b}$$

Remark 2.1: Different from the definition in [7], [8], a horizon N is introduced in Definition 2.2. This horizon captures the time length that the state can be kept in $\mathbb{Q}$ with a given probability α under admissible control inputs. In particular, the case of the finite horizon N survives the stabilization of the system (1) when the support set $\mathbb{W}$ is unbounded despite the bounded control inputs, which is beyond the scope of the traditional stochastic stabilization.

In the above definition, the set $\mathbb{Q}$ is called the target set while the set $\mathbb{P}$ is called the region of attraction. The objective of this paper is to characterize these two sets satisfying conditions (2b)-(2a) for the given probability levels α, β and the required horizon N. If $\mathbb{Q} = \mathbb{P} = \mathbb{R}^{n_x}$, then the system (1) is always $(\mathbb{R}^{n_x}, \mathbb{R}^{n_x}, N, \alpha, \beta)$-stabilizable in probability for any $\alpha, \beta \in (0, 1]$ and $N \in \bar{\mathbb{N}}$. To avoid triviality, we assume that $\mathbb{Q} \subseteq \mathbb{R}^{n_x}$ is compact. In this paper, we first solve the following problem.

Problem 2.1: Consider the system (1). Given $\alpha \in (0, 1]$ and $N \in \bar{\mathbb{N}}$, determine a compact set $\mathbb{Q} \in \mathcal{B}(\mathbb{X})$ such that the condition (2a) holds.

After characterizing the set $\mathbb{Q}$, we solve Problem 2.2.

Problem 2.2: Consider the system (1). Given $\beta \in (0, 1]$ and the target set $\mathbb{Q}$, determine a set $\mathbb{P} \in \mathcal{B}(\mathbb{X})$ such that the condition (2b) holds.

III. Characterization of Target Set

This section provides an answer to Problem 2.1. We remark that the target set $\mathbb{Q}$ satisfying (2a) is the N-step probabilistic controlled invariant set in [9], [10]. Introduce the probability with which the state x_k will remain within $\mathbb{Q}$ for all $k \in \mathbb{N}_{[0,N]}$ under the policy $\boldsymbol{\mu}$:

$$p^{\boldsymbol{\mu}}_{N,\mathbb{Q}}(x_0) = \Pr\{\forall k \in \mathbb{N}_{[0,N]}, x_k \in \mathbb{Q}\}.$$

The condition (2a) is equivalent to $\forall x_0 \in \mathbb{Q}$,

$$p^*_{N,\mathbb{Q}}(x_0) \triangleq \sup_{\boldsymbol{\mu} \in \mathcal{M}} p^{\boldsymbol{\mu}}_{N,\mathbb{Q}}(x_0) \geq \alpha.$$

To solve Problem 2.1, we consider two cases: $N < \infty$ and $N = \infty$.

A. Case 1: $N < \infty$

When $N \in \mathbb{N}$, the optimization problem $\sup_{\boldsymbol{\mu} \in \mathcal{M}} p^{\boldsymbol{\mu}}_{N,\mathbb{Q}}(x_0)$ is a finite-horizon stochastic optimal control problem, which can be solved by a dynamic program.

Lemma 3.1: [11], [12] Define the value function $V^*_{k,\mathbb{Q}} : \mathbb{X} \to [0, 1], k \in \mathbb{N}_{[0,N]}$, by the backward recursion:

$$V^*_{k,\mathbb{Q}}(x) = \sup_{u \in \mathbb{U}} \mathbf{1}_{\mathbb{Q}}(x) \int_{\mathbb{Q}} V^*_{k+1,\mathbb{Q}}(y) f(y - Ax - Bu) dy, \tag{3}$$

with initialization $V^*_{N,\mathbb{Q}}(x) = 1, x \in \mathbb{Q}$. Then, $p^*_{N,\mathbb{Q}}(x_0) = V^*_{0,\mathbb{Q}}(x)$. The optimal Markov policy $\boldsymbol{\mu}^*_{\mathbb{Q}} = (\mu^*_{0,\mathbb{Q}}, \mu^*_{1,\mathbb{Q}}, \ldots, \mu^*_{N-1,\mathbb{Q}})$ exists and is given by $\forall x \in \mathbb{Q}$,

$$\mu^*_{k,\mathbb{Q}}(x) = \arg\sup_{u \in \mathbb{U}} \mathbf{1}_{\mathbb{Q}}(x) \int_{\mathbb{Q}} V^*_{k+1,\mathbb{Q}}(y) f(y - Ax - Bu) dy.$$

Proposition 3.1: For the system (1) and a compact set $\mathbb{Q} \in \mathcal{B}(\mathbb{X})$, we have that $p^*_{N,\mathbb{Q}}(x) \geq \alpha$, $\forall x \in \mathbb{Q}$, if $\forall x \in \mathbb{Q}$, there exists a $u \in \mathbb{U}$ such that

$$\int_{\mathbb{Q}} f(y - Ax - Bu) dy \geq \alpha^{\frac{1}{N}}. \tag{4}$$

Proof: The fact that $p^*_{N,\mathbb{Q}}(x) \geq \alpha$, $\forall x \in \mathbb{Q}$, is equivalent to that $V^*_{0,\mathbb{Q}}(x) \geq \alpha$, $\forall x \in \mathbb{Q}$. Let us prove by induction that

$$V^*_{k,\mathbb{Q}}(x) \geq \alpha^{\frac{N-k}{N}}, \ \forall x \in \mathbb{Q}. \tag{5}$$

From Lemma 3.1, (5) holds when $k = N$. Assume that (5) holds for $k+1$, $k \in \mathbb{N}_{[0,N-1]}$. Then, by (3) and (4), it is easy to show that (5) holds for k. That is, $V^*_{0,\mathbb{Q}}(x) \geq \alpha$, $\forall x \in \mathbb{Q}$. The proof is completed. ∎

Corollary 3.1: Consider a scalar system

$$x_{k+1} = ax_k + bu_k + w_k, \tag{6}$$

where $x_k, u_k, w_k \in \mathbb{R}$, $a, b \neq 0$, $w_k \sim \mathcal{N}(0, \sigma^2)$, and $\mathbb{U} = \{u \in \mathbb{R} \mid |u| \leq \bar{u}\}$ with $0 < \bar{u} < \infty$. For a set $\mathbb{Q} = \{x \in \mathbb{R} \mid |x| \leq \bar{x}\}$ with $\bar{x} > 0$, the fact that $p^*_{N,\mathbb{Q}}(x) \geq \alpha$, $\forall x \in \mathbb{Q}$, holds if there exists a u satisfying $|u| \leq \bar{u}$ such that

$$\frac{1}{2}(\mathrm{erf}(\frac{(1-a)\bar{x} - bu}{\sigma\sqrt{2}}) - \mathrm{erf}(\frac{-(1+a)\bar{x} - bu}{\sigma\sqrt{2}})) \geq \alpha^{\frac{1}{N}}, \quad (7)$$

where $\mathrm{erf}(\cdot)$ is the error function.

Proof: Since $w_k \sim \mathcal{N}(0, \sigma^2)$, x_k and u_k, $x_{k+1} \sim \mathcal{N}(ax_k + bu_k, \sigma^2)$. The condition (4) is equivalent to $\forall x$ such that $|x| \leq \bar{x}$, there exists a $|u| \leq \bar{u}$ such that

$$\frac{1}{2}(\mathrm{erf}(\frac{\bar{x} - (ax+bu)}{\sigma\sqrt{2}}) - \mathrm{erf}(\frac{-\bar{x} - (ax+bu)}{\sigma\sqrt{2}})) \geq \alpha^{\frac{1}{N}}. \quad (8)$$

Let $z = ax + bu$ and $g(z) = \frac{1}{2}(\mathrm{erf}(\frac{\bar{x}-z}{\sigma\sqrt{2}}) - \mathrm{erf}(\frac{-\bar{x}-z}{\sigma\sqrt{2}}))$. Taking derivative with respect to z, we have

$$\frac{dg}{dz} = -\frac{1}{\sigma\sqrt{2\pi}}(\exp(-(\frac{\bar{x}-z}{\sigma\sqrt{2}})^2) - \exp(-(\frac{-\bar{x}-z}{\sigma\sqrt{2}})^2)).$$

Considering the monotonicity of the functions $\exp(-x)$ and x^2, we have that $g(z)$ is increasing when $z \leq 0$ and is decreasing when $z \geq 0$.

Let us prove that $\forall x$ such that $|x| \leq \bar{x}$, there exists a u with $|u| \leq \bar{u}$ such that (8) holds if for any $x \in \{\bar{x}, -\bar{x}\}$, there exists a u with $|u| \leq \bar{u}$ such that (8) holds. Due to the symmetry, we only prove the above statement for $x = \bar{x}$. Assume that when $x = \bar{x}$, there exists a u^* with $|u^*| \leq \bar{u}$ such that (8) holds. For any all $x \in [0, \bar{x}]$, let $u(x) = \frac{u^* x}{\bar{x}}$. Since $|\frac{x}{\bar{x}}| \leq 1$, $|u(x)| \leq |u^*| \leq \bar{u}$. Furthermore, if $a\bar{x} + bu^* \geq 0$,

$$0 \leq ax + bu(x) = \frac{(a\bar{x} + bu^*)x}{\bar{x}} \leq a\bar{x} + bu^*.$$

Due to the monotonicity of $g(z)$, we have

$$g(ax + bu(x)) \geq g(a\bar{x} + bu^*) \geq \alpha^{\frac{1}{N}}.$$

On the other hand, if $a\bar{x} + bu^* \leq 0$,

$$0 \geq ax + bu(x) = \frac{(a\bar{x} + bu^*)x}{\bar{x}} \geq a\bar{x} + bu^*.$$

It follows that

$$g(ax + bu(x)) \geq g(a\bar{x} + bu^*) \geq \alpha^{\frac{1}{N}}.$$

The proof is completed. ■

B. Case 2: $N = \infty$

When $N = \infty$, the problem $\sup_{\boldsymbol{\mu} \in \mathcal{M}} p^{\boldsymbol{\mu}}_{\infty,\mathbb{Q}}(x_0)$ is an infinite-horizon stochastic optimal control problem.

Lemma 3.2: Define the value function $G^*_{k,\mathbb{Q}} : \mathbb{X} \to [0,1], k \in \mathbb{N}_{\geq 0}$, in the forward recursion:

$$G^*_{k+1,\mathbb{Q}}(x) = \sup_{u \in \mathbb{U}} \mathbf{1}_{\mathbb{Q}}(x) \int_{\mathbb{Q}} G^*_{k,\mathbb{Q}}(y) f(y - Ax - Bu) dy, \quad (9)$$

initialized with $G^*_{0,\mathbb{Q}}(x) = 1, x \in \mathbb{Q}$. Then, for all $x \in \mathbb{Q}$, the limit $G^*_{\infty,\mathbb{Q}}(x)$ exists and satisfies

$$G^*_{\infty,\mathbb{Q}}(x) = \sup_{u \in \mathbb{U}} \mathbf{1}_{\mathbb{Q}}(x) \int_{\mathbb{Q}} G^*_{\infty,\mathbb{Q}}(y) f(y - Ax - Bu) dy,$$

and $p^*_{\infty,\mathbb{Q}}(x) = G^*_{\infty,\mathbb{Q}}(x)$. Furthermore, an optimal stationary Markov policy $\boldsymbol{\mu}^* = (\bar{\mu}^*_{\mathbb{Q}}, \bar{\mu}^*_{\mathbb{Q}}, \ldots)$ exists and is given by

$$\bar{\mu}^*_{\mathbb{Q}}(x) = \arg\sup_{u \in \mathbb{U}} \mathbf{1}_{\mathbb{Q}}(x) \int_{\mathbb{Q}} G^*_{\infty,\mathbb{Q}}(y) f(y - Ax - Bu) dy.$$

Proposition 3.2: For the system (1) and a compact set $\mathbb{Q} \in \mathcal{B}(\mathbb{X})$ and $\alpha \in (0,1]$, $p^*_{\infty,\mathbb{Q}}(x) \geq \alpha$, $\forall x \in \mathbb{Q}$, hold only if the support set $\mathbb{W}$ is bounded.

Proof: From (9), we have that $\forall x \in \mathbb{Q}$,

$$0 \leq G^*_{\infty,\mathbb{Q}}(x) \leq \cdots \leq G^*_{k,\mathbb{Q}}(x) \leq \cdots \leq G^*_{1,\mathbb{Q}}(x) = 1.$$

Define $\lambda = \sup_{x \in \mathbb{Q}} \sup_{u \in \mathbb{U}} \int_{\mathbb{Q}} f(y - Ax - Bu) dy$. If the support set $\mathbb{W}$ is unbounded, we have $0 \leq \lambda < 1$. Define a new sequence $\hat{G}^*_{k,\mathbb{Q}} = \sup_{x \in \mathbb{Q}} G^*_{k,\mathbb{Q}}(x)$. Then, it follows that

$$\hat{G}^*_{k+1,\mathbb{Q}} \leq \lambda \hat{G}^*_{k,\mathbb{Q}}, \text{ and } \hat{G}^*_{\infty,\mathbb{Q}} = \lim_{k \to \infty} \hat{G}^*_{k,\mathbb{Q}} = 0,$$

which contradicts to the condition (2a) where $\alpha > 0$. The proof is completed. ■

Proposition 3.3: [10] For the system (1) and a compact set $\mathbb{Q} \in \mathcal{B}(\mathbb{X})$, $p^*_{\infty,\mathbb{Q}}(x) \geq \alpha$, $\forall x \in \mathbb{Q}$, hold if there exits $\mathbb{Q}_f \in \mathcal{B}(\mathbb{X})$ with $\mathbb{Q}_f \subseteq \mathbb{Q}$ such that

$$\text{(i)} \quad \forall x \in \mathbb{Q}_f, \exists u \in \mathbb{U}, \int_{\mathbb{Q}_f} f(y - Ax - Bu) dy = 1, \quad (10a)$$

$$\text{(ii)} \quad \forall x \in \mathbb{Q} \setminus \mathbb{Q}_f, \exists u \in \mathbb{U}, \int_{\mathbb{Q}_f} f(y - Ax - Bu) dy \geq \alpha. \quad (10b)$$

Proof: Similar to Theorem 3 in [10]. ■

Corollary 3.2: Consider the scalar system (6) with $w_k \sim \mathcal{U}(-\bar{w}, \bar{w})$ and $\mathbb{U} = \{u \in \mathbb{R} \mid |u| \leq \bar{u}\}$, where $0 < \bar{w} < \infty$ and $0 < \bar{u} < \infty$. For a set $\mathbb{Q} = \{x \in \mathbb{R} \mid |x| \leq \bar{x}\}$, the fact that $p^*_{N,\mathbb{Q}}(x) \geq \alpha$, $\forall x \in \mathbb{Q}$, holds if

(i) $\bar{x} \geq |\frac{b}{a}|\bar{u} \geq \bar{w}$;

(ii) there exists a u with $|u| \leq \bar{u}$ such that $\frac{\bar{l} - \underline{l}}{2\bar{w}} \geq \alpha$, where $\bar{l} = (a\bar{x} + bu + \bar{w}) \wedge (|\frac{b}{a}|\bar{u})$ and $\underline{l} = (a\bar{x} + bu - \bar{w}) \vee (-|\frac{b}{a}|\bar{u})$

Proof: If the condition (i) holds, we have that there exists a set $\mathbb{Q}_f = \{x \in \mathbb{R} \mid |x| \leq |\frac{b}{a}|\bar{u}\}$ such that $\mathbb{Q}_f \subseteq \mathbb{Q}$ and $\forall x \in \mathbb{Q}_f$, there exists a u such that $ax + bu = 0$ and

$$\int_{\mathbb{Q}_f} f(y - Ax - Bu) dy = \int_{-|\frac{b}{a}|\bar{u}}^{|\frac{b}{a}|\bar{u}} \frac{1}{2\bar{w}} dy \geq \int_{-\bar{w}}^{\bar{w}} \frac{1}{2\bar{w}} dy = 1.$$

That is, the condition (10a) in Proposition 3.3 holds.

Note that $\mathbb{Q} \setminus \mathbb{Q}_f = [-\bar{x}, -|\frac{b}{a}|\bar{u}) \cup (\frac{b}{a}|\bar{u}|, \bar{x}]$. Following the similar idea in the proof of Corollary 3.1, we can show that the condition (ii) in Proposition 3.3 holds if when $x = \bar{x}$, there exists a u with $|u| \leq \bar{u}$ such that (10b) holds and when $x = -\bar{x}$, there exists a u with $|u| \leq \bar{u}$ such that (10b) holds. This is equivalent to the condition (ii).

The proof is completed. ■

IV. Characterization of Region of Attraction

After computing the target sets, let us consider Problem 2.2, i.e., how to characterize the region of attraction. We will provide two methods: one is based on the solution to the stochastic optimal first-entry time problem while the other is based on stochastic backward reachable sets.

A. Characterization by solving a stochastic optimal first-entry time problem

For the system (1) and a horizon $T \in \mathbb{N}$, given an initial state x_0 and a Markov policy $\boldsymbol{\mu} \in \mathcal{M}$, we can generate a stochastic process $\{X_k(x_0, \boldsymbol{\mu}, T)\}_{k=0}^T$ with $X_0 = x_0$.

Definition 4.1: Given a set $\mathbb{Q} \in \mathcal{B}(\mathbb{X})$, the first entry time to $\mathbb{Q}$ of the stochastic process $\{X_k(x_0, \boldsymbol{\mu})\}_{k=0}^T$ is defined as

$$\tau(x_0, \boldsymbol{\mu}, T, \mathbb{Q}) = \inf\{j \geq 0 \mid X_j(x_0, \boldsymbol{\mu}, T) \in \mathbb{Q}\}.$$

Let us define the functions $J : \mathbb{R}^{n_x} \to [0, 1]$ as

$$J(x_0, T, \mathbb{Q}) = \sup_{\boldsymbol{\mu} \in \mathcal{M}} \mathrm{E}\{\mathbf{1}_{\mathbb{Q}}(X_{\hat{\tau}}(x_0, \boldsymbol{\mu}, T))\} \tag{11}$$

where $\hat{\tau} = \tau(x_0, \boldsymbol{\mu}, T, \mathbb{Q}) \wedge T$.

Lemma 4.1: [13], [14] Define the value function $F_{k,\mathbb{Q}} : \mathbb{X} \to [0, 1], k \in \mathbb{N}_{[0,T]}$, by the backward recursion:

$$F^*_{k,\mathbb{Q}}(x) = \sup_{u \in \mathbb{U}}\{\mathbf{1}_{\mathbb{X}\setminus\mathbb{Q}}(x) \int_{\mathbb{Q}} F^*_{k+1,\mathbb{Q}}(y) f(y - Ax - Bu)dy + \mathbf{1}_{\mathbb{Q}}(x)\},$$

with initialization $F^*_{T,\mathbb{Q}}(x) = 1, x \in \mathbb{Q}$. Then, $J(x, T, \mathbb{Q}) = F^*_{0,\mathbb{Q}}(x)$. The optimal Markov policy $\boldsymbol{\mu}^*_{\mathbb{Q}} = (\mu^*_{0,\mathbb{Q}}, \mu^*_{1,\mathbb{Q}}, \ldots, \mu^*_{T-1,\mathbb{Q}})$ exists and is given by

$$\mu^*_{k,\mathbb{Q}}(x) = \arg\sup_{u \in \mathbb{U}}\{\mathbf{1}_{\mathbb{X}\setminus\mathbb{Q}}(x) \int_{\mathbb{Q}} F^*_{k+1,\mathbb{Q}}(y) f(y - Ax - Bu)dy + \mathbf{1}_{\mathbb{Q}}(x)\},\ x \in \mathbb{Q}, k \in \mathbb{N}_{[0,T-1]}.$$

Recall the condition (2b). Given $\beta \in (0, 1]$, define the set

$$\mathbb{S}(T, \beta, \mathbb{Q}) = \{x_0 \in \mathbb{R}^{n_x} \mid J(x_0, T, \mathbb{Q}) \geq \beta\}. \tag{12}$$

Proposition 4.1: Given a target set $\mathbb{Q}$, the region of attraction $\mathbb{P}$ satisfying (2b) is

$$\mathbb{P} = \bigcup_{T \in \mathbb{N}} \mathbb{S}(T, \beta, \mathbb{Q}) \tag{13}$$

Proof: The condition (2b) is equivalent to that $\forall x \in \mathbb{P}$, there exists a $T \in \mathbb{N}$ and a corresponding policy $\boldsymbol{\mu}$ such that $\Pr(\exists k \in \mathbb{N}_{[0,T]}, x_k \in \mathbb{Q}) \geq \beta$. Then, the result (13) directly follows from the definitions of (11) and (12). ∎

B. Characterization by stochastic backward reachable sets

This subsection will provide an overapproximation of the region of attraction by using stochastic backward reachable sets. For the system (1) and time horizon $T \in \mathbb{N}$, given an initial state x_0 and a sequence of control inputs $\{u_k\}_{k=0}^{T-1}$, it follows that

$$x_k = A^k x_0 + \sum_{i=0}^{k-1} A^{k-1-i} B u_i + \sum_{i=0}^{k-1} A^{k-1-i} w_i. \tag{14}$$

Definition 4.2: Given a set $\mathbb{Q} \in \mathcal{B}(\mathbb{X})$ and $\beta \in (0, 1]$, the T-step β-stochastic backward reachable set $\mathbb{Y}(T, \beta, \mathbb{Q})$ to $\mathbb{Q}$ is defined by

$$\mathbb{Y}(T, \beta, \mathbb{Q}) = \{x_0 \in \mathbb{R}^{n_x} \mid \exists u_i \in \mathbb{U}, i \in \mathbb{N}_{[0,T-1]}, \Pr\{x_T \in \mathbb{Q}\} \geq \beta\}, \tag{15}$$

where x_T is defined in (14). The β-stochastic backward reachable set $\tilde{\mathbb{Y}}(\beta, \mathbb{Q})$ to $\mathbb{Q}$ is defined by

$$\tilde{\mathbb{Y}}(\beta, \mathbb{Q}) = \bigcup_{T \in \mathbb{N}} \mathbb{Y}(T, \beta, \mathbb{Q}).$$

Lemma 4.2: For the sets $\mathbb{S}(T, \beta, \mathbb{Q})$ in (12) and $\mathbb{Y}(T, \beta, \mathbb{Q})$ in (15), we have

$$\mathbb{S}(T, \beta, \mathbb{Q}) \subseteq \bigcup_{k \in \mathbb{N}_{[0,T]}} \mathbb{Y}(k, \beta, \mathbb{Q}).$$

Proof: Define the set $\mathcal{U} = \mathbb{U} \times \cdots \mathbb{U} \triangleq \mathbb{U}^\infty$. Then, the result directly follows that $\mathcal{M} \subseteq \mathcal{U}$. ∎

Proposition 4.2: Given a target set $\mathbb{Q}$, the β-stochastic backward reachable set $\tilde{\mathbb{Y}}(\beta, \mathbb{Q})$ provides an overapproximation to the region of attraction $\mathbb{P}$ satisfying (2b), i.e., $\mathbb{P} \subseteq \tilde{\mathbb{Y}}(\beta, \mathbb{Q})$.

Proof: The result directly follows from Lemma 4.2. ∎

Let us consider how to compute the T-step β-stochastic backward reachable set $\mathbb{Y}(T, \beta, \mathbb{Q})$. For simplicity, let us assume the target set $\mathbb{Q}$ is a polyhedron of the form

$$\mathbb{Q} = \{x \in \mathbb{R}^{n_x} \mid Hx \leq h\},$$

where H and h are with appropriate dimensions. Then, the set $\mathbb{Y}(T, \beta, \mathbb{Q})$ can be rewritten as

$$\mathbb{Y}(T, \beta, \mathbb{Q}) = \{x_0 \in \mathbb{R}^{n_x} \mid H(A^k x_0 + \sum_{i=0}^{T-1} A^{T-1-i} B u_i) + \tilde{h} \leq h,\ \forall i \in \mathbb{N}_{[0,T-1]}, u_i \in \mathbb{U}\}$$

where $\tilde{h}$ can be computed by a chance-constrained optimization problem

$$\begin{aligned} &\min\ \mathbf{1}^T \tilde{h} \\ &\text{subject to } \Pr\{\sum_{i=0}^{k-1} H A^{k-1-i} w_i \leq \tilde{h}\} = \beta. \end{aligned}$$

Please refer to [15], [16] for the numerical methods to solve the above chance constrained program.

Corollary 4.1: Consider the scalar system (6) with $w_k \sim \mathcal{N}(0, \sigma^2)$ and $\mathbb{U} = \{u \in \mathbb{R} \mid |u| \leq \bar{u}\}$, where $0 < \bar{u} < \infty$. For a set $\mathbb{Q} = \{x \in \mathbb{R} \mid |x| \leq \bar{x}\}$, the T-step β-stochastic backward reachable set $\mathbb{Y}(T, \beta, \mathbb{Q})$ to $\mathbb{Q}$ is nonempty if there exists $\tilde{x}_T \in \mathbb{R}$ with $\tilde{x}_T \geq 0$ and a sequence of $|u_i| \leq \bar{u}$, $\forall i \in \mathbb{N}_{[0,T-1]}$ such that

$$\frac{1}{2}(\mathrm{erf}(\frac{\bar{x} - x_T}{\sqrt{2\tilde{\sigma}_T}}) - \mathrm{erf}(\frac{-\bar{x} - x_T}{\sqrt{2\tilde{\sigma}_T}})) \geq \beta. \tag{16}$$

where $x_T = a^T \tilde{x}_T + \sum_{i=0}^{T-1} a^{T-1-i} b u_i$ and $\tilde{\sigma}_T = \sum_{i=0}^{T-1} a^{2i} \sigma^2$. Furthermore, $[-\tilde{x}_T, \tilde{x}_T] \subseteq \mathbb{Y}(T, \beta, \mathbb{Q})$.

Proof: The results mainly follows from the fact that

$$x_k \sim \mathcal{N}(a^T \tilde{x}_0 + \sum_{i=0}^{k-1} a^{k-1-i} b u_i, \tilde{\sigma}_k).$$

Then, we can easily derive (16) by translating the definition of T-step β-stochastic backward reachable set. The result $[-\tilde{x}_T, \tilde{x}_T] \subseteq \mathbb{Y}(T, \beta, \mathbb{Q})$ can be proved by following the similar idea of the proof in Corollary 3.1. ∎

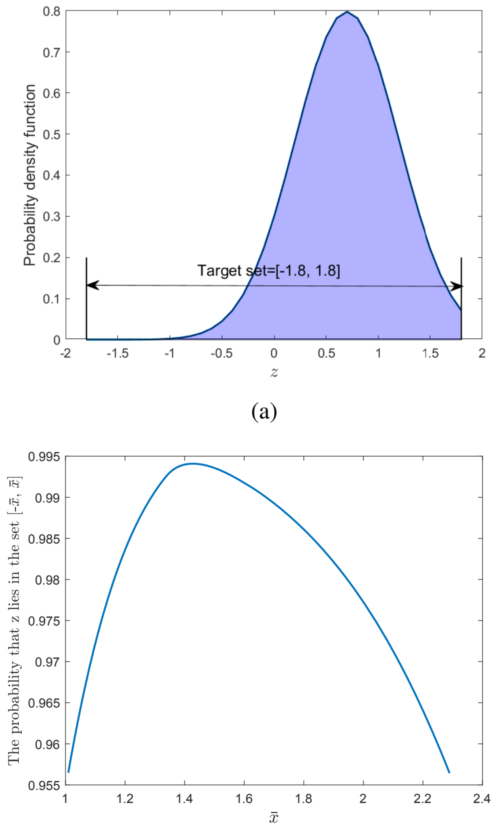

Fig. 1: (a) Probability density function of $z \sim \mathcal{N}(0.70, 0.25)$ over $\mathbb{Q} = [-1.8, 1.8]$; (b) the probability that $z(\bar{x}) \sim \mathcal{N}(\bar{z}, 0.25)$ lies in $\mathbb{Q} = [-\bar{x}, \bar{x}]$.

V. Examples

Consider the scalar system (6) with $b = 1$, $w_k \sim \mathcal{N}(0, \sigma^2)$, and $\mathbb{U} = \{u \in \mathbb{R} \mid |u| \leq 2\}$, where $0 < \bar{u} < \infty$. We aim to characterize the target set $\mathbb{Q}$ and the region of attraction $\mathbb{P}$ for the above system such that this system is $(\mathbb{P}, \mathbb{Q}, N, \alpha, \beta)$-stabilizable in probability.

A. Target set $\mathbb{Q}$

We will first determine the target set $\mathbb{Q}$ by Corollary 3.1. Let $a = 1.5$ in this following.

- Let $\sigma = 0.5$, $N = 5$ and $\alpha = 0.8$. Consider a set $\mathbb{Q} = [-1.8, 1.8]$. In this case, we have $1.8a - b\bar{u} = 0.70$. Let $z \sim \mathcal{N}(0.70, 0.25)$ and its probability density function over set $\mathbb{Q}$ is shown in Fig. 1(a). Since the probability that z lies in the set $\mathbb{Q}$ is 0.9861, greater than $\alpha^{\frac{1}{5}} = 0.9564$, we have $\mathbb{Q} = [-1.8, 1.8]$ is a target set which satisfies the condition (7).
  Consider a family of sets $\mathbb{Q} = [-\bar{x}, \bar{x}]$, where $\bar{x} \in [1.0087, 2.2902]$. Let $z(\bar{x}) \sim \mathcal{N}(\bar{z}, 0.25)$ where $\bar{z} = \begin{cases} 0, & \text{if } \bar{x} <= |\frac{b}{a}|\bar{u} \\ a\bar{x} + b\bar{u}, & \text{otherwise} \end{cases}$. The corresponding probability that $z(\bar{x})$ lies in $\mathbb{Q} = [-\bar{x}, \bar{x}]$ is shown in Fig. 1(b) and is always greater than $\alpha^{\frac{1}{5}} = 0.9564$. Hence, we conclude that for any $\bar{x} \in [1.0087, 2.2902]$, the set $\mathbb{Q} = [-\bar{x}, \bar{x}]$ is a target set. Furthermore, we can show that $\mathbb{Q} = [-1.0087, 1.0087]$ is the smallest target set satisfying (7) while $\mathbb{Q} = [-2.2902, 2.2902]$ is the largest target set satisfying (7).
- Note that the characterization of target set depends on the parameters σ, N, and α. Figs. 2(a)–(c) show how the largest target set scales with such parameters. We can see that the largest target set becomes smaller by choosing larger σ, N, and α.

B. Region of attraction $\mathbb{P}$

Let us characterize the region of attraction $\mathbb{P}$ by the stochastic backward reachable sets. Set $\sigma = 0.5$, $N = 5$, and $\alpha = 0.8$. We can show that the set $\mathbb{Q} = [-1.8, 1.8]$ is a target set for all $a \in [0.5, 1.5]$.

- Let $\beta = 0.80$ and $a = 1$. The T-step β-stochastic backward reachable sets are shown in Fig. 3(a). The maximal T such that the T-step β-stochastic backward reachable set is nonempty is 7. So the the overapproximation of the region of attraction is $\mathbb{Y}(7, \beta, \mathbb{Q}) = [-14.4661, 14.4661]$. Fig. 4 shows state trajectories starting from $x_0 = 14.4661$ under 1000 realization of disturbances, where the green part is the region of attraction and the yellow part is the target set. Under 1000 realizations, 80.3% (close to β) of the state trajectories enter the target set at time step $k = 7$ and then they can stay in the target set for 5 steps with probability greater than $\alpha = 80\%$.
- Note that the characterization of region of attraction depends on the parameters a and β. According to Corollary 4.1, Figs. 3(b)–(c) show how the overapproximation of the region of attraction scales with the parameters a and β. The region of attraction becomes smaller when choosing larger a and β. In addition, if $|a| \leq 1$, the region of attraction is the whole real space $\mathbb{R}$.

VI. Conclusion

This paper proposed a new notion of stabilization in probability for discrete-time stochastic systems and studied the characterizations of region of attraction and target set used in this definition. Sufficient conditions were derived such that a compact set is a target set with given horizon and probability level. Furthermore, given a target set, two methods were used to characterize the region of attraction. One was based on the solution of the stochastic optimal first-entry time problem while the other was based on stochastic backward reachable sets. Simulations were given to illustrate the effectiveness of the theoretical results.

One future direction is to apply the proposed definition of stabilization in probability to safety-critical control and stochastic predictive control.

VII. Acknowledgement

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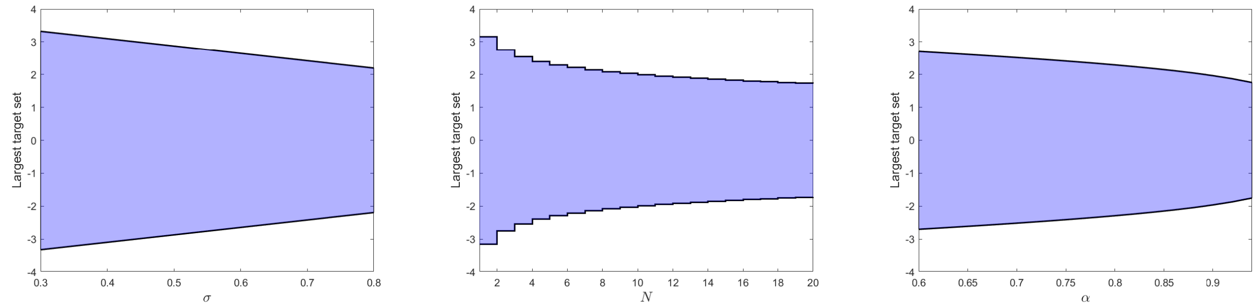
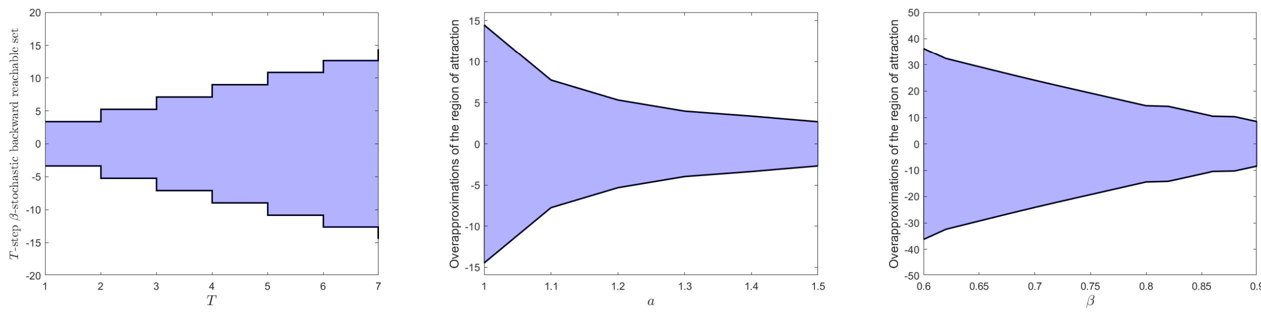


(a) (b) (c)

Fig. 2: The evolution of largest target set over (a) the standard variance σ when setting $N = 5$ and $\alpha = 0.5$; (b) the horizon N when setting $\alpha = 0.5$ and $\sigma = 0.5$; (c) the probability level α when setting $N = 5$ and $\sigma = 0.5$.



(a) (b) (c)

Fig. 3: (a) The T-step β-stochastic backward reachable set; (b) the overapproximations of the region of attraction along the system parameter a when setting $\beta = 0.8$; (c) the overapproximations of the region of attraction along the system parameter β when setting $a = 1$.

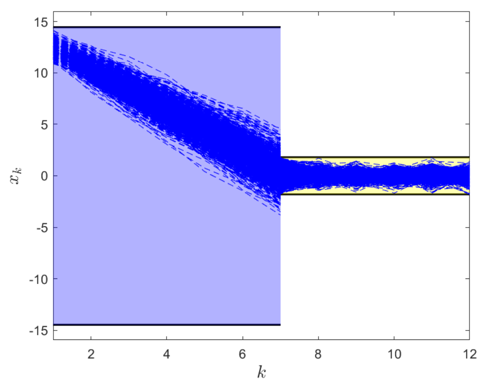


Fig. 4: State trajectories under 1000 realization of disturbances.

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