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Further results on saturated globally stabilizing linear state feedback control laws for single-input neutrally stable planar systems

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***Abstract*— It is known that for single-input neutrally stable planar systems, there exists a class of saturated globally stabilizing linear state feedback control laws. The goal of this paper is to characterize the dynamic behavior for such a system under arbitrary locally stabilizing linear state feedback control laws. On the one hand, for the continuous-time case, we show that all locally stabilizing linear state feedback control laws are also globally stabilizing control laws. On the other hand, for the discrete-time case, we first show that this property does not hold by explicitly constructing nontrivial periodic solution for a particular system. We then show for an example that there exists more globally stabilizing linear state feedback control laws than well known ones in the literature.**

I. Introduction

Linear systems subject to actuator saturation are common in almost every physical application and have been the subject of extensive study, e.g., [1], [2]. Internal stabilization for this class of systems has a long history. It was established in e.g., [3]–[5] that global stabilization of linear systems subject to actuator saturation can be achieved if and only if the system is asymptotically null controllable with bounded controls (ANCBC), i.e., the linear system in the absence of actuator saturation is stabilizable, and has all its open-loop poles in the closed left-half plane in the continuous-time setting and strictly within or on the unit circle in the discrete-time setting. The seminal work of [6] established that a chain of integrator with order equal to or greater than three cannot be globally asymptotically stabilized by any saturated linear static state feedback control law. The results implies that for achieving global stability of the closed-loop system, one has to design nonlinear feedback control laws. Teel [7] proposed a nonlinear combination of saturation functions of linear state feedback control laws that globally stabilizes a chain of integrators of arbitrary order.

There is limited knowledge regarding which linear systems subject to actuator saturation allow for global stabilization via linear state feedback control laws. In the continuous-time setting, it is known that any linear static state feedback law which locally stabilizes the double integrator[4], also globally stabilizes the system in the presence of actuator saturation, e.g., [8]–[12]. For open-loop neutrally stable[5] systems, it is known that there exist linear state feedback control laws which globally stabilize systems in the presence of actuator saturation, e.g., [13], [14]. Some extensions to continuous-time linear systems consisting of a mixture of neutrally stable dynamics and double integrators have been established in [8], [15], where the authors designed globally stabilizing linear state feedback control laws for this class of systems. Recently, for continuous-time single-input planar ANCBC systems in the controllable canonical form, [12] showed that the saturated linear state feedback control laws whose feedback gains are negative globally stabilize such a system.

The goal of our on-going research is to identify which locally stabilizing linear state feedback control laws yield globally asymptotic stability of the closed-loop system in the presence of actuator saturation and which ones do not. In a recent paper [16], for the discrete-time double integrator, we have established that the class of locally stabilizing linear state feedback control laws splits into two parts. Although one part yields global asymptotic stability of the closed-loop system, the other part does not yield global stability of the closed-loop system, which was shown by explicitly constructing nontrivial periodic solutions.

This paper may be viewed as a further step towards our goal. Here we consider single-input neutrally stable planar systems. On the one hand, for continuous-time systems, we show that any locally stabilizing linear state feedback control law is also a globally stabilizing control law. On the other hand, for discrete-time systems, we first show that this property does not hold by providing a particular example for which the closed-loop system exhibit nontrivial periodic solutions. We then present for another example and show that there exists another globally stabilizing linear state feedback control law other than well known ones in the literature.

II. Problem formulation

Consider a neutrally stable system subject to actuator saturation described by

$$\rho x = Ax + B\sigma(u), \tag{1}$$

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[4]Note that a linear state feedback law with arbitrary negative feedback gains stabilizes the double integrator in the absence of actuator saturation.

[5]A linear system is said to be open-loop neutrally stable if, for a continuous-time system, all its open-loop poles are in the closed left-half complex plane with those on the imaginary axis being simple, or for a discrete-time system, all its open-loop poles are strictly within or on the unit circle with those on the unit circle being simple.

where ρ is an operator indicating the time derivative $\frac{d}{dt}$ for continuous-time systems and a forward unit time shift for discrete-time systems, $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the input, and $\sigma(u) = \left[\sigma_1(u_1), \ldots, \sigma_1(u_m)\right]^{\mathrm{T}}$, where each $\sigma_1(u_i)$ is the standard saturation function $\sigma_1(u_i) = \operatorname{sgn}(u_i)\min\{1, |u_i|\}$.

Assumption 1: The pair (A,B) is stabilizable.

Under Assumption 1, in a suitable basis we have:

$$A = \begin{bmatrix} A_c & 0 \\ 0 & A_s \end{bmatrix}, \quad B = \begin{bmatrix} B_c \\ B_s \end{bmatrix},$$

where A_c is such that $A_c + A_c^{\mathrm{T}} = 0$ for continuous-time systems, and $A_c^{\mathrm{T}} A_c = I$ for discrete-time systems, A_s is asymptotically stable, and the pair (A_c, B_c) is controllable. Without loss of generality, we can ignore the asymptotically stable subsystem, and make the following assumption.

Assumption 2: The pair (A,B) is controllable, $A + A^{\mathrm{T}} = 0$ for continuous-time systems, and $A^{\mathrm{T}} A = I_n$ for discrete-time systems.

We note that for the system (1) which satisfies Assumption 2, controllability of the pair (A,B) is equivalent to stabilizability of the pair (A,B).

As mentioned in the introduction, it is known that if the system is open-loop neutrally stable and stabilizable, then there exist globally stabilizing linear state feedback control laws in the presence of actuator saturation. The above result is recapped in the following lemma.

Lemma 1: Consider the system given by (1), and let Assumption 2 hold. Then

- for continuous-time systems, the linear state feedback control law
$$u = -\kappa B^{\mathrm{T}} x, \tag{2}$$
where $\kappa > 0$ globally stabilizes the system (1).
- for discrete-time systems, the linear state feedback control law
$$u = -\kappa B^{\mathrm{T}} A x, \tag{3}$$
where $\kappa > 0$ and $\kappa B^{\mathrm{T}} B < 2 I_m$, globally stabilizes the system (1).

Our goal is to characterize the dynamic behavior of the system with other locally stabilizing linear state feedback control laws, that is, the ones which are not of the form (2) for continuous-time systems or not of the form (3) for discrete-time systems. As a first step, this paper considers the planar system with a single input.

III. Continuous-time case

In the continuous-time setting, the planar system under Assumption 2 can be represented by the following dynamical equation

$$\dot{x} = Ax + B\sigma(u), \tag{4}$$

where

$$A = \begin{bmatrix} 0 & b \\ -b & 0 \end{bmatrix}, \text{ and } B = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix},$$

with $b, b_1, b_2 \in \mathbb{R}$ such that the pair (A,B) is controllable, which is equivalent to $b \neq 0$ and $b_1^2 + b_2^2 \neq 0$.

We will consider the linear state feedback control law of the form

$$u = Fx := f_1 x_1 + f_2 x_2. \tag{5}$$

Remark 1: The planar system (4) under Assumption 2 can always be transformed into a controllable canonical form via a suitable state transformation, i.e.,

$$\dot{\tilde{x}} = \begin{bmatrix} 0 & 1 \\ -b^2 & 0 \end{bmatrix} \tilde{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \sigma(u). \tag{6}$$

It is clear that the linear state feedback control law

$$u = \begin{bmatrix} \tilde{f}_1 & \tilde{f}_2 \end{bmatrix} \tilde{x}, \tag{7}$$

with feedback gains $\tilde{f}_1 < b^2$ and $\tilde{f}_2 < 0$ locally stabilizes the system (6). Also note that it follows from [12, Theorem 1] that the linear state feedback control law (7) with feedback gains $\tilde{f}_1 < 0$ and $\tilde{f}_2 < 0$ globally stabilizes such a system. The following theorem shows that arbitrary locally stabilizing linear state feedbacks also globally stabilize such a system.

Theorem 1: Consider the system given by (4). Any locally stabilizing linear state feedback control law, i.e., (5) with F chosen such that $A + BF$ is Hurwitz stable, also globally stabilizes the system (4) in the presence of actuator saturation.

Proof: Without loss of generality, we assume that the system (4) has the following form,

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + B\sigma(u), \tag{8}$$

where

$$A = \begin{bmatrix} 0 & b \\ -b & 0 \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

since if this is not the case, we can always find a non-singular state transformation $\bar{x} := T_s x$, where

$$T_s = \frac{1}{b_1^2 + b_2^2} \begin{bmatrix} b_2 & -b_1 \\ b_1 & b_2 \end{bmatrix},$$

such that $\bar{A} = T_s A T_s^{-1}$ and $\bar{B} = T_s B$ are of the form

$$\bar{A} = \begin{bmatrix} 0 & b \\ -b & 0 \end{bmatrix} \text{ and } \bar{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

From Routh-Hurwitz criteria, we see that in order to achieve the local stability of the closed-loop system of (8) and (5), it is required that

$$b(b - f_1) > 0, \quad f_2 < 0. \tag{9}$$

The condition (9) is equivalent to the following conditions:

- Case I: $b > 0$, in this case, $f_1 < b$ and $f_2 < 0$,
- Case II: $b < 0$, in this case, $f_1 > b$ and $f_2 < 0$.

From Lemma 1, we know that the control laws (5) with $f_1 = 0$ and any $f_2 < 0$ globally stabilize the system (8). Hence, we assume $f_1 \neq 0$ in the rest of the proof.

Consider the following Lyapunov candidate

$$V = \tfrac{|b|}{2}(x_1^2 + x_2^2) - \tfrac{\operatorname{sgn}(b)}{f_1} \int_0^{f_1 x_1} \sigma(y)\, dy. \tag{10}$$

We shall only consider Case I, i.e., $b > 0$ since Case II where $b < 0$ then follows easily due to symmetry. Let us first show that V defined by (10) is positive definite. For Case I, we consider two different cases:

Case I.a $f_1 < 0$. In this case, note that

$$2\int_0^{f_1x_1} \sigma(y)dy = 2\sigma(f_1x_1)f_1x_1 - \sigma^2(f_1x_1) \geq \sigma^2(f_1x_1).$$

Therefore,

$$\begin{aligned} V &= \tfrac{b}{2}(x_1^2+x_2^2) - \tfrac{1}{f_1}\int_0^{f_1x_1} \sigma(y)dy \\ &\geq \tfrac{b}{2}(x_1^2+x_2^2) - \tfrac{1}{2f_1}\sigma^2(f_1x_1) \geq 0, \end{aligned}$$

where we have used the fact that $f_1 < 0$. Moreover, $V = 0$ if and only if $x_1 = x_2 = 0$.

Case I.b: $0 < f_1 < b$. In this case, we show that V is positive definite by considering three cases:

Case I.b.1: $f_1x_1 > 1$, in this case, we have

$$\begin{aligned} V &= \tfrac{b}{2}(x_1^2+x_2^2) - \tfrac{1}{f_1}(f_1x_1 - \tfrac{1}{2}) \\ &= \tfrac{b}{2}(x_1 - \tfrac{1}{b})^2 + \tfrac{b}{2}x_2^2 + \tfrac{b-f_1}{2f_1b} > 0, \end{aligned}$$

where we have used the fact that $0 < f_1 < b$.

Case I.b.2: $|f_1x_1| \leq 1$, in this case, we have

$$V = \tfrac{b}{2}(x_1^2+x_2^2) - \tfrac{f_1}{2}x_1^2 = \tfrac{b-f_1}{2}x_1^2 + \tfrac{b}{2}x_2^2 \geq 0,$$

where we have used the fact that $0 < f_1 < b$, and the equality holds if and only if $x_1 = x_2 = 0$.

Case I.b.3: $f_1x_1 < -1$, in this case, we have

$$\begin{aligned} V &= \tfrac{b}{2}(x_1^2+x_2^2) + \tfrac{1}{f_1}(f_1x_1 + \tfrac{1}{2}) \\ &= \tfrac{b}{2}(x_1 + \tfrac{1}{b})^2 + \tfrac{b}{2}x_2^2 + \tfrac{b-f_1}{2bf_1} > 0, \end{aligned}$$

where we have used the fact that $0 < f_1 < b$.

Hence we have shown that $V \geq 0$ and $V = 0$ if and only if $x_1 = x_2 = 0$, thus, V is positive definite. Next, let us show that $\dot{V} \leq 0$. With some algebra, we obtain that

$$\begin{aligned} \dot{V} &= bx_1\dot{x}_1 + bx_2\dot{x}_2 - \sigma(f_1x_1)\dot{x}_1 \\ &= bx_2[\sigma(f_1x_1 + f_2x_2) - \sigma(f_1x_1)]. \end{aligned}$$

Let $f_1x_1 + f_2x_2 = v_1$ and $f_1x_1 = v_2$, then we get

$$\dot{V} = \tfrac{b}{f_2}(v_1 - v_2)[\sigma(v_1) - \sigma(v_2)]. \tag{11}$$

Note that $(v_1 - v_2)[\sigma(v_1) - \sigma(v_2)] \geq 0$ since $\sigma(\cdot)$ is a non-decreasing function. Then $\dot{V} \leq 0$ since $f_2 < 0$ and $b > 0$. Moreover, $\dot{V} = 0$ if and only if

- Case 1: $v_1 > 1$ and $v_2 > 1$,
- Case 2: $|v_1| \leq 1$, $|v_2| \leq 1$, and $v_1 = v_2$,
- Case 3: $v_1 < -1$ and $v_2 < -1$.

We shall show that the trajectory cannot stay for Case 1 and Case 3. Let us only consider the Case 1, since Case 3 then follows easily due to the symmetric property. Assume that there exists a trajectory $x(t)$ for all $t \geq 0$ which stays in Case 1, then for all $t \geq 0$, the following inequalities hold:

$$\begin{aligned} f_1x_1(t) &> 1, \\ f_1x_1(t) + f_2x_2(t) &> 1. \end{aligned}$$

We then consider two cases depending on the value of f_1:

Case 1.a: $f_1 < 0$, in this case, we have $x_1(t) < 0$ for all $t \geq 0$ since $f_1x_1(t) > 1$. Therefore $\dot{x}_2 = -bx_1 + \sigma(f_1x_1 + f_2x_2) > 0$, and thus there exists a finite time $T_1 > 0$, such that $\dot{x}_1(t) = bx_2(t) > 0$ for $t \geq T_1$. This yields that eventually $x_1(t) > 0$, that is, there exists a finite time $T_2 > 0$ such that $x_1(t) > 0$ for $t \geq T_2$, which contradicts with the fact that $x_1(t) < 0$ for all $t \geq 0$.

Case 1.b: $0 < f_1 < b$, in this case, we have $x_1(t) > \frac{1}{b}$ for all $t \geq 0$ since $f_1x_1(t) > 1$. Therefore $\dot{x}_2 = -bx_1 + \sigma(f_1x_1 + f_2x_2) < 0$, and thus there exists a finite time $T_3 > 0$, such that $\dot{x}_1(t) = bx_2(t) < 0$ for $t \geq T_3$. This yields that eventually $x_1(t) < 0$, that is, there exists a finite time $T_4 > 0$ such that $x_1(t) < 0$ for $t \geq T_4$, which contradicts with the fact that $x_1(t) > \frac{1}{b}$ for all $t \geq 0$.

Therefore, there exists no trajectory which can stay in Case 1 and Case 3. By continuity of the trajectory, we see that the trajectory whose initial condition satisfy the conditions of Case 1 or Case 3 will eventually move to the region where $|v_1| \leq 1$ and $|v_2| \leq 1$. In order to keep $\dot{V} = 0$ in this region, we need that $v_1(t) = v_2(t)$ for all $t \geq 0$, i.e., Case 2. It is easy to see that $x_2(t) \equiv 0$ for all $t \geq 0$ since $f_2x_2(t) = v_1(t) - v_2(t) = 0$ and $f_2 < 0$. Also note that $\dot{x}_2(t) \equiv 0$ for all $t \geq 0$. However,

$$\dot{x}_2(t) = -bx_1(t) + \sigma(v_1(t)) = (f_1 - b)x_1(t).$$

Hence $x_1(t) \equiv 0$ for $t \geq 0$ since $f_1 < b$. Therefore, the only trajectory which can stay in Case 2 is the origin. The global asymptotic stability of the closed-loop system of the system (4) and (5) where f_1 and f_2 satisfy (9) and $f_1 \neq 0$ then follows from LaSalle's Invariance Principle. ■

IV. Discrete-time case

In Section III, we have shown that for continuous-time single-input neutrally stable planar systems, any locally stabilizing linear static state feedback also globally stabilizes the system in the presence of actuator saturation. In this section, we first investigate whether this nice property still holds in the discrete-time setting. The following theorem shows that this property does not hold in general by considering a particular example.

Theorem 2: Consider the following system

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} \frac{3}{5} & \frac{4}{5} \\ -\frac{4}{5} & \frac{3}{5} \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \sigma(u(k)), \tag{12}$$

and a linear static state feedback control law

$$u(k) = Fx(k) := f_1x_1(k) + f_2x_2(k). \tag{13}$$

There exist feedback gains f_1 and f_2 which locally stabilize the system (12), however, do not globally stabilize the system (12) in the presence of actuator saturation.

Proof: From Jury's test [17], we see that any feedback control law (13), where f_1 and f_2 satisfy the following conditions

$$2f_1 + f_2 - 2 < 0 \tag{14a}$$

$$2f_2 - f_1 + 4 > 0, \tag{14b}$$

$$-10 < 3f_2 - 4f_1 < 0, \tag{14c}$$

$$|(1 + \tfrac{3}{5}f_2 - \tfrac{4}{5}f_1)^2 - 1| > |(\tfrac{6}{5} + f_2)(\tfrac{3}{5}f_2 - \tfrac{4}{5}f_1)|, \tag{14d}$$

stabilizes the system (12) in the absence of actuator saturation. Note that the condition (14d) is automatically satisfied if the conditions (14a), (14b) and (14c) are satisfied. This result is illustrated by Fig. 1. That is, whenever the feedback gains f_1 and f_2 take their values within the triangular ABC, the closed-loop system is locally stable; otherwise unstable.

We shall prove the theorem by establishing non-zero periodic solution with period of $T = 4$ for a subset of locally stabilizing linear state feedbacks. The periodic solution with period of $T = 4$ that we will construct is such that the system is always in saturation, and the saturated input sequence is composed of 1 for first two steps, followed by -1 for the next two steps. In order to have $x(4) = x(0)$, from the dynamical equation (12), we must have that $x_1(0) = \frac{2}{3}$ and $x_2(0) = -\frac{4}{3}$.

Clearly, this will yield the required periodic solution if $x(0)$, f_1, and f_2 are such that $u(k) \geq 1$ for $k = 0,1$ and $u(k) \leq -1$ for $k = 2,3$, i.e.,

$$\begin{aligned} Fx(0) &\geq 1, \\ F(Ax(0)+B) &\geq 1, \\ F(A^2x(0)+AB+B) &\leq -1, \\ F(A^3x(0)+A^2B+AB-B) &\leq -1. \end{aligned}$$

Plugging the initial condition $x(0) = [\frac{2}{3}, -\frac{4}{3}]^{\mathrm{T}}$ into the above equations yields

$$\begin{aligned} \tfrac{2}{3}f_1 - \tfrac{4}{3}f_2 &\geq 1, \\ -\tfrac{2}{3}f_1 - \tfrac{1}{3}f_2 &\geq 1, \end{aligned}$$

which is a subset of f_1 and f_2 satisfying the Jury's condition (14), i.e., triangular ADE in Fig. 1. ■

Remark 2: It follows from Lemma 1 that the system (12) is globally stabilized via linear state feedback control law (13) with $f_1 = \frac{4k}{5}$ and $f_2 = -\frac{3k}{5}$ where $0 < k < 2$, i.e., the line BF in Fig. 1.

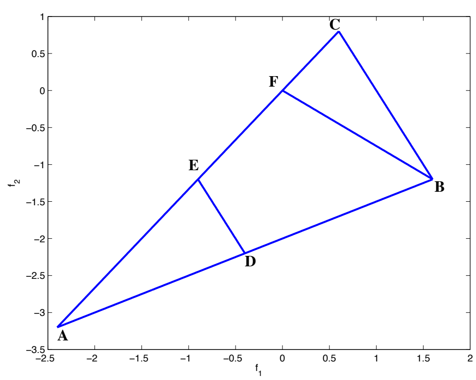

Fig. 1. Stability characteristics as a function of f_1 and f_2

We have seen in the proof of Theorem 2 that the closed-loop system of (12) and (13) has a periodic solution with period of 4 if the feedback gains are chosen within the triangular ADE in Fig. 1. This is illustrated in the following example.

Example 1: Consider the system (12) with a linear state feedback control law (13), where $f_1 = -1.6$ and $f_2 = -2.5$, and the initial condition $x(0) = [\frac{2}{3}, -\frac{4}{3}]^{\mathrm{T}}$. In Fig. 2, we clearly see the period orbit with period $T = 4$.

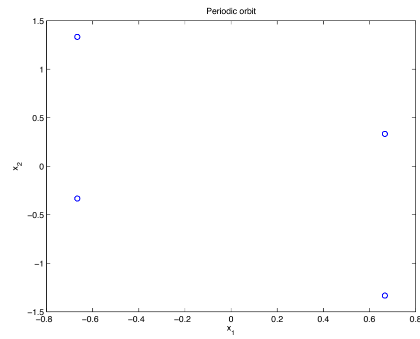

Fig. 2. Periodic orbit of period 4

Theorem 2 provides an example which shows that for discrete-time single-input neutrally stable planar systems, unlike in the continuous-time setting, not all locally stabilizing linear state feedbacks globally stabilize the system in the presence of actuator saturation by explicit constructing nontrivial periodic solution. However, the question whether there exist more locally stabilizing linear state feedbacks than well known ones of the form (3) presented in Lemma 1 also globally stabilize the discrete-time system in the presence of actuator saturation still remains open. The rest of this section answers this question by considering the following example.

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \sigma(u(k)). \tag{15}$$

Remark 3: It follows from Lemma 1 that a family of linear state feedback control laws

$$u(k) = -\kappa B^{\mathrm{T}} Ax(k) = \kappa x_1(k), \tag{16}$$

where $0 < \kappa < 2$ globally stabilize the system (15) in the presence of actuator saturation.

The following theorem shows that for the system (15), there exists another globally stabilizing linear state feedback control law, which is not of the form (16).

Theorem 3: The linear state feedback

$$u(k) = f_1 x_1(k) + f_2 x_2(k) := x_1(k) + \frac{1}{2} x_2(k). \tag{17}$$

globally stabilizes the system (15).

Proof: Consider the following Lyapunov candidate

$$V = \tfrac{1}{2}(x_1^2 + x_2^2) - \int_0^{x_1} \sigma(y) dy. \tag{18}$$

We first note that V defined by (18) is positive semidefinite, which follows from the analysis for the positive definiteness of V defined by (10) with $b = 1$ and $f_1 = 1$ in Theorem 1 for the case $0 < f_1 < b$. Also note that $V = 0$ if and only if $|x_1| \leq 1$ and $x_2 = 0$.

Next, let us show that $\Delta V(k) = V(k+1) - V(k) \leq 0$. With some algebra, we obtain that

$$\begin{aligned} V(k+1) &= \tfrac{1}{2}[x_1^2(k+1) + x_2^2(k+1)] - \int_0^{x_1(k+1)} \sigma(s)ds \\ &= \tfrac{1}{2}[x_2^2 + (-x_1 + \sigma(u))^2] - \tfrac{1}{2}[2\sigma(x_2)x_2 - \sigma^2(x_2)]. \end{aligned}$$

Therefore,

$$\begin{aligned} \Delta V(k) = \tfrac{1}{2}[-2x_1\sigma(u) + \sigma^2(u)] - \tfrac{1}{2}[2\sigma(x_2)x_2 - 2\sigma(x_1)x_1 \\ + \sigma^2(x_1) - \sigma^2(x_2)]. \end{aligned}$$

We will show that $\Delta V \leq 0$ by partitioning $\mathbb{R}^2$ into 12 regions. Due to the symmetric property, we can only consider 6 regions, and show that $\Delta V \leq 0$ in each region.

Region 1: $R_1 = \{(x_1, x_2) \in \mathbb{R}^2 | x_1 \geq 1, x_2 \geq 1\}$. Clearly, in this case, we have $u = f_1x_1 + f_2x_2 = x_1 + \frac{1}{2}x_2 > 1$. Therefore,

$$\Delta V = \tfrac{1}{2}(-2x_1 + 1) - \tfrac{1}{2}(2x_2 - 2x_1) = -x_2 + \tfrac{1}{2} < -\tfrac{1}{2} < 0.$$

Region 2: $R_2 = \{(x_1, x_2) \in \mathbb{R}^2 | x_1 \geq 1, x_2 \leq -1\}$. For this case, we need further partition R_2 into three regions depending on whether $\sigma(u)$ is saturated or not.

Region $2a$: $R_{2a} = \{(x_1, x_2) \in \mathbb{R}^2 | x_1 \geq 1, x_2 \leq -1, x_1 + \frac{1}{2}x_2 \geq 1\}$. In this case, we get

$$\Delta V = \tfrac{1}{2}(-2x_1 + 1) - \tfrac{1}{2}(-2x_2 - 2x_1) = x_2 + \tfrac{1}{2} < -\tfrac{1}{2} < 0.$$

Region $2b$: $R_{2b} = \{(x_1, x_2) \in \mathbb{R}^2 | x_1 \geq 1, x_2 \leq -1, |x_1 + \frac{1}{2}x_2| \leq 1\}$. In this case, we get

$$\begin{aligned} \Delta V &= -x_1(x_1 + \tfrac{1}{2}x_2) + \tfrac{1}{2}(x_1 + \tfrac{1}{2}x_2)^2 + x_2 + x_1 \\ &= -\tfrac{1}{2}x_1^2 + \tfrac{1}{8}x_2^2 + x_1 + x_2. \end{aligned}$$

Since $\frac{d\Delta V}{dx_1} = -x_1 + 1 \leq 0$, thus ΔV decreases as x_1 increases. Hence, ΔV attains its maximum for minimal x_1. We then further partition Region 2b as follows:

Case $2b.1$: The minimal x_1 is equal to 1, that is $-4 < x_2 < -1$. we then get that

$$\Delta V_{\max} = \tfrac{1}{8}(x_2 + 4)^2 - \tfrac{3}{2} < -\tfrac{3}{8} < 0.$$

Case $2b.2$: The minimal x_1 is attain at $x_1 = -\frac{1}{2}x_2 - 1$, we then obtain that $\Delta V_{\max} = -\frac{3}{2} < 0$.

Region $2c$: $R_{2c} = \{(x_1, x_2) \in \mathbb{R}^2 | x_1 \geq 1, x_2 \leq -1, x_1 + \frac{1}{2}x_2 \leq -1\}$. In this case, we get

$$\Delta V = \tfrac{1}{2}(2x_1 + 1) - \tfrac{1}{2}(-2x_2 - 2x_1) = 2x_1 + x_2 + \tfrac{1}{2} < -\tfrac{3}{2} < 0.$$

Region 3: $R_3 = \{(x_1, x_2) \in \mathbb{R}^2 | x_1 \geq 1, |x_2| \leq 1\}$. For this case, we need further partition R_3 into two regions depending on whether $\sigma(u)$ is saturated or not.

Region $3a$: $R_{3a} = \{(x_1, x_2) \in \mathbb{R}^2 | x_1 \geq 1, |x_2| \leq 1, x_1 + \frac{1}{2}x_2 \geq 1\}$. In this case, we get

$$\Delta V = \tfrac{1}{2}(-2x_1 + 1) - \tfrac{1}{2}(2x_2^2 - 2x_1 + 1 - x_2^2) = -\frac{1}{2}x_2^2 \leq 0,$$

where the equality holds if and only if $x_2 = 0$.

Region $3b$: $R_{3b} = \{(x_1, x_2) \in \mathbb{R}^2 | x_1 \geq 1, |x_2| \leq 1, 0 \leq x_1 + \frac{1}{2}x_2 \leq 1\}$. In this case, we get

$$\begin{aligned} \Delta V &= -x_1(x_1 + \tfrac{1}{2}x_2) + \tfrac{1}{2}(x_1 + \tfrac{1}{2}x_2)^2 \\ &\quad - \tfrac{1}{2}(2x_2^2 - 2x_1 + 1 - x_2^2) \\ &= -\tfrac{1}{2}x_1^2 - \tfrac{3}{8}x_2^2 + x_1 - \tfrac{1}{2}. \end{aligned}$$

Since $\frac{d\Delta V}{dx_1} = -x_1 + 1 \leq 0$, thus ΔV decreases as x_1 increases. Hence, ΔV attains its maximum for minimal x_1, that is $x_1 = 1$. We then obtain that $\Delta V_{\max} = -\frac{3}{8}x_2^2 \leq 0$, where the equality holds if and only if $x_2 = 0$.

Region 4: $R_4 = \{(x_1, x_2) \in \mathbb{R}^2 | 0 \leq x_1 \leq 1, x_2 \geq 1\}$. For this case, we need further partition it into two regions depending on whether $\sigma(u)$ is saturated or not.

Region $4a$: $R_{4a} = \{(x_1, x_2) \in \mathbb{R}^2 | 0 \leq x_1 \leq 1, x_2 \geq 1, x_1 + \frac{1}{2}x_2 \geq 1\}$. In this case, we get

$$\begin{aligned} \Delta V &= \tfrac{1}{2}(-2x_1 + 1) - \tfrac{1}{2}(2x_2 - 2x_1^2 + x_1^2 - 1) \\ &= \tfrac{1}{2}x_1^2 - x_1 - x_2 + 1 \\ &= \tfrac{1}{2}(x_1 - 1)^2 - x_2 + \tfrac{1}{2} \leq -x_2 + 1 < 0, \end{aligned}$$

where we have used that $(x_1 - 1)^2 \leq 1$.

Region $4b$: $R_{4b} = \{(x_1, x_2) \in \mathbb{R}^2 | 0 \leq x_1 \leq 1, x_2 \geq 1, 0 \leq x_1 + \frac{1}{2}x_2 \leq 1\}$. In this case, we get

$$\begin{aligned} \Delta V &= -x_1(x_1 + \tfrac{1}{2}x_2) + \tfrac{1}{2}(x_1 + \tfrac{1}{2}x_2)^2 - \tfrac{1}{2}(2x_2 - x_1^2 - 1) \\ &= \tfrac{1}{8}(x_2 - 4)^2 - \tfrac{3}{2} \leq -\tfrac{3}{8} < 0, \end{aligned}$$

where we have used that $1 < x_2 < 2$ to get the second last inequality.

Region 5: $R_5 = \{(x_1, x_2) \in \mathbb{R}^2 | 0 \leq x_1 \leq 1, x_2 \leq -1\}$.

For this case, we need further partition R_5 into two regions depending on whether $\sigma(u)$ is saturated or not.

Region $5a$: $R_{5a} = \{(x_1, x_2) \in \mathbb{R}^2 | 0 \leq x_1 \leq 1, x_2 \leq -1, x_1 + \frac{1}{2}x_2 \leq -1\}$. In this case, we get

$$\begin{aligned} \Delta V &= \tfrac{1}{2}(2x_1 + 1) - \tfrac{1}{2}(-2x_2 - 2x_1^2 + x_1^2 - 1) \\ &= \tfrac{1}{2}x_1^2 + x_1 + x_2 + 1. \end{aligned}$$

Since $\frac{d\Delta V}{dx_1} = x_1 + 1 > 0$, thus ΔV decreases as x_1 decreases. Hence, ΔV attains its maximum for maximal x_1. We then further partition Region 5a as follows:

Region $5a1$: The maximal x_1 is attains at $x_1 = -\frac{1}{2}x_2 - 1$, that is $-4 < x_2 < -2$, we then obtain that $\Delta V_{\max} = \frac{1}{8}(x_2 + 4)^2 - \frac{3}{2} < -1 < 0$.

Region $5a2$: The maximal x_1 is equal to 1, that is $x_2 \leq -4$, we then get that

$$\Delta V_{\max} = x_2 + \tfrac{5}{2} \leq -\tfrac{3}{2} < 0.$$

Region $5b$: $R_{5b} = \{(x_1, x_2) \in \mathbb{R}^2 | 0 \leq x_1 \leq 1, x_2 \geq 1, 0 \leq x_1 + \frac{1}{2}x_2 \leq 1\}$. In this case, we get

$$\begin{aligned} \Delta V &= -x_1(x_1 + \tfrac{1}{2}x_2) + \tfrac{1}{2}(x_1 + \tfrac{1}{2}x_2)^2 \\ &\quad - \tfrac{1}{2}(2x_2 - 2x_1^2 + x_1^2 - 1) \\ &= \tfrac{1}{8}(x_2 + 4)^2 - \tfrac{3}{2} < -\tfrac{3}{8} < 0, \end{aligned}$$

where we have used that $-4 < x_2 < -1$ to get the second last inequality.

Region 6: $R_6 = \{(x_1, x_2) \in \mathbb{R}^2 | 0 \leq x_1 \leq 1, -1 \leq x_2 \leq 1\}$. For this case, we need further partition R_6 into two regions depending on whether $\sigma(u)$ is saturated or not.

Region $6a$: $R_{6a} = \{(x_1, x_2) \in \mathbb{R}^2 | 0 \leq x_1 \leq 1, -1 \leq x_2 \leq 1, x_1 + \frac{1}{2}x_2 \geq 1\}$. In this case, we get

$$\begin{aligned} \Delta V &= \tfrac{1}{2}(-2x_1 + 1) - \tfrac{1}{2}(2x_2^2 - 2x_1^2 + x_1^2 - x_2^2) \\ &= \tfrac{1}{2}x_1^2 - \tfrac{1}{2}x_2^2 - x_1 + \tfrac{1}{2}. \end{aligned}$$

Note that $\frac{d\Delta V}{dx_1} = x_1 - 1 \leq 0$. Therefore, ΔV attains its maximum for minimal x_1, that is, $x_1 = 1 - \frac{1}{2}x_2$, we then obtain that $\Delta V_{\max} = -\frac{3}{8}x_2^2 \leq 0$, where the equality holds if and only if $x_2 = 0$.

Region $6b$: $R_{6b} = \{(x_1, x_2) \in \mathbb{R}^2 | 0 \leq x_1 \leq 1, -1 \leq x_2 \leq 1, |x_1 + \frac{1}{2}x_2| \leq 1\}$. In this case, we get

$$\begin{aligned}\Delta V &= -x_1(x_1 + \tfrac{1}{2}x_2) + \tfrac{1}{2}(x_1 + \tfrac{1}{2}x_2)^2 - \tfrac{1}{2}(x_2^2 - x_1^2) \\ &= -\tfrac{3}{8}x_2^2 \leq 0,\end{aligned}$$

where the equality holds if and only if $x_2 = 0$.

The above six regions are visualized by Fig. 3. Due to symmetry, we have omitted the symmetric counterparts of these six regions.

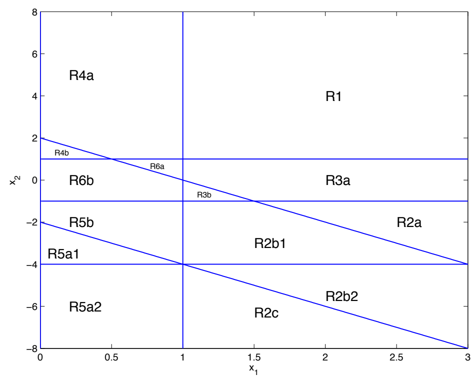

Fig. 3. Part of the partition for $\mathbb{R}^2$

Hence, we have shown that for the above six regions $\Delta V \leq 0$, and $\Delta V = 0$ if and only if $x_1 \geq 0$ and $x_2 = 0$. Again due to symmetry, we conclude that $\Delta V \leq 0$ and $\Delta V = 0$ if and only if $x_2 = 0$. From (15), it is easy to see that in order to keep $x_2(k) \equiv 0$ for all $k \geq 0$, we need $|x_1| \leq 1$. Hence, solutions which can stay in the set $\{(x_1, x_2) \in \mathbb{R}^2 | x_2 = 0\}$ must stay in the set $Z := \{(x_1, x_2) | |x_1| \leq 1, x_2 = 0\}$. Since $\Delta V \leq 0$, we conclude that $V(k)$ is non-increasing. Thus, $\lim_{k\to\infty} V(k) = V_*$ for some $V_* \geq 0$. This implies that $\Delta V(k) \to 0$ as $k \to \infty$ and hence $x(k) \to Z$ as $k \to \infty$ as shown above. This implies that

$$x(k+1) = \begin{bmatrix} x_2(k) \\ -x_1(k) + \sigma(x_1(k) + \frac{1}{2}x_2(k)) \end{bmatrix} \to 0 \text{ as } k \to \infty.$$

Hence, we have shown that the closed-loop system of (15) and (17) is globally attractive. Then global asymptotic stability of the closed-loop system then follows from the locally stability of the closed-loop system. ∎

V. Conclusion

This paper considers single-input neutrally stable planar systems via saturated linear state feedback control laws. On the one hand, we show that for continuous-time systems, any locally stabilizing linear state feedback also globally stabilizes the system in the presence of actuator saturation. On the other hand, for discrete-time systems, we show this property does not hold in general and there exist more globally stabilizing linear state feedback than well known ones in the literature. This is a further step towards our future goal, that is, to identify which locally stabilizing linear state feedback control laws yield globally asymptotic stability of the closed-loop system in the presence of actuator saturation and which ones do not.

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