Using Piecewise-Constant Congestion Taxing Policy in Repeated Routing Games*

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Abstract

We consider repeated routing games with piecewise-constant congestion taxing in which a central planner sets and announces the congestion taxes for fixed windows of time in advance. Specifically, congestion taxes are calculated using marginal congestion pricing based on the flow of the vehicles on each road prior to the beginning of the taxing window. The piecewise-constant taxing policy in motivated by that users or drivers may dislike fast-changing prices and that they also prefer prior knowledge of the prices. We prove that the multiplicative update rule converges to a socially optimal flow when using vanishing step sizes. Considering that the algorithm cannot adapt itself to a changing environment when using vanishing step sizes, we propose using constant step sizes in this case. Then, however, we can only prove the convergence of the dynamics to a neighborhood of the socially optimal flow (with its size being of the order of the selected step size).

1 Introduction

Urban traffic congestion is the source of many problems in large metropolitan areas, such as increased travel times and fuel consumption, air pollution, and dampened economic growth [1, 13, 14]. To circumvent some of these problems, local governments in several cities, such as Stockholm, London, San Francisco, and
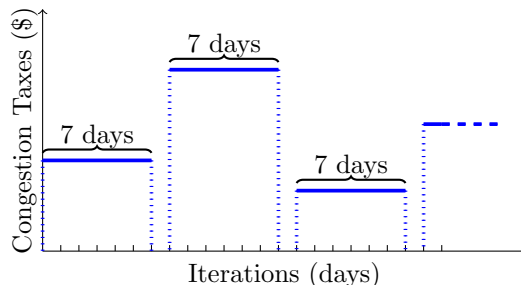

Figure 1: An illustrative example of the piecewise-constant congestion taxing policy.

Singapore, have introduced various congestion taxing[1] schemes [5, 11, 17, 25]. However, there are several issues that should be addressed in congestion taxing systems. For example, the charges need to be adapted because of the traffic flow increases due to external factors, primarily increasing population in the county [5]. More importantly, the implemented fixed tolls do not react to temporary traffic changes and are designed based on the average behaviour of the travellers [27]. To avoid this problem, dynamic congestion pricing techniques have been employed [9]. However, imposing dynamic taxes is certainly controversial or, to say the least, cumbersome to understand for the drivers as they need to calculate and respond to time-varying congestion taxes at the same time as driving. Therefore, it is desirable to devise a slowly-varying or piecewise-constant congestion charges for the roads (that are announced well in advance so that the drivers can respond to them properly).

In this paper, we propose a piecewise-constant congestion tax policy for repeated routing games in which groups of drivers use the transportation network on a daily basis. We specifically use the multiplicative update rule [15] for updating the flows on various paths. We assume that the central planner sets the congestion taxes for wide windows of iterations in advance and announce the taxes publicly for those days. This would amount to piecewise-constant congestion taxes as the

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[1]We use the terms "toll", "congestion tax", "congestion price", and "congestion charge" interchangeably throughout the paper.

tolls stay constant for a number of days (e.g., a week, a month). Figure 1 shows an illustrative example of such piecewise-constant congestion taxes when the congestion taxes gets updated on a weekly basis. Our interest in this scheme is motivated by the facts that the drivers dislike fast-changing prices and want prior knowledge of the prices. The proposed congestion taxes are calculated using marginal congestion prices based on the flow of the vehicles on each road prior to the beginning of the taxing window. We prove that, for the proposed piecewise-constant congestion taxes, the multiplicative update rule converges to a socially optimal decision if its step sizes is set to be of the order of $1/k$ for iteration k. Unfortunately, the shrinking step size renders the algorithms impractical for the cases where the parameters of the routing game (e.g., the demands) vary over time since the proposed dynamics cannot adapt fast enough, especially, after a long time, because the step size becomes very small. Following this observation, we propose using a constant step size. Doing so, we realize that we can only converge to a neighborhood of the socially optimal flow, with its size being proportional to the selected step size. This is an interesting trade-off because so long as the step size is large the algorithm can adapt rapidly to the changes in the routing game; however, the solution can potentially be far from the socially optimal flow.

Repeated routing games have attracted attention recently [3, 4, 16]. For instance, in [3], the authors studied no-regret learning (i.e., the difference between the average latency caused by the online decisions and the average latency for the best fixed decision in hindsight grows very slowly). They also proved the convergence of a subsequence of the flows to a neighbourhood of the equilibrium. The convergence result was further strengthened to the whole sequence of flows in [16]. Repeated routing games are in close connection with evolutionary game theory [12,23,24,28], in which users adopt simple update rules motivated by biological systems and evolutionary observations, e.g., the users meet with a given probability with other users and replicate their behaviour if it results in a better utility. To the best of our knowledge, none of these studies propose a piecewise constant scheme for setting congestion taxes in repeated routing games to achieve a socially optimal flow.

The rest of the paper is organized as follows. In Section 2, we introduce our notations for routing game and review some results in this area. We present our results in the repeated routing game in Section 3. Numerical examples are presented in Section 4. Finally, we finish the paper with the conclusions and avenues for future research in Section 5.

2 Routing Games

In what follows, we use $\mathbb{R}$ and $\mathbb{N}$ to denote the sets of real and integer numbers, respectively. We also define $\mathbb{R}_{\geq(>)a} = \{x \in \mathbb{R} | x \geq (>)a\}$ for any $a \in \mathbb{R}$. Furthermore, let $\llbracket K \rrbracket = \{1, \ldots, K\}$ for any $K \in \mathbb{N}$.

We model the transportation network with a directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ in which $\mathcal{V}$ denotes the nodes in the network (e.g., intersections) and $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ denotes the edges in the network (e.g., roads). We assume that the graph can admit parallel edges. We are also provided with a set of source–destination nodes $\{(s_k, d_k)\}_{k \in \llbracket K \rrbracket}$, $K \in \mathbb{N}$, where each pair (s_k, d_k), $k \in \llbracket K \rrbracket$, should transfer a total flow of $F_k \in \mathbb{R}_{>0}$. The assumption that $F_k \neq 0$ is without loss of generality as, otherwise, we can remove the source–destination nodes with zero flow from the problem without changing the problem. Let $\mathcal{P}_k$ denote the set of all directed paths that connect the source s_k to destination d_k for any $k \in \llbracket K \rrbracket$, where a directed path from node s_k to node d_k is an ordered sequence of edges $((i_j, i_{j+1}))_{j=1}^{n-1} \in \mathcal{E}^n$ such that $i_1 = s_k$ and $i_n = d_k$. Moreover, let us define the set of all paths as $\mathcal{P} = \cup_{k \in \llbracket K \rrbracket} \mathcal{P}_k$.

We use $f_p \in \mathbb{R}_{\geq 0}$ to denote the flow of vehicles on a path $p \in \mathcal{P}$. In addition, we define the aggregate flow vector $f = (f_p)_{p \in \mathcal{P}} \in \mathbb{R}^{|\mathcal{P}|}$. A flow vector is called feasible if $\sum_{p \in \mathcal{P}_k} f_p = F_k$ for all $k \in \llbracket K \rrbracket$. Let us denote the set of all such feasible flows with $\mathcal{F}((F_k)_{k \in \llbracket K \rrbracket})$. When the source–destination flows $(F_k)_{k \in \llbracket K \rrbracket}$ can be deduced from the context or are irrelevant to the discussion, with slight abuse of notation, we shorten the notation to $\mathcal{F}$. For any aggregate vector of path flows $(f_p)_{p \in \mathcal{P}}$, we can define edge flows[2] $\phi_e = \sum_{p \in \mathcal{P}: e \in p} f_p \in \mathbb{R}_{\geq 0}$ for all $e \in \mathcal{E}$. We make the following standing assumption.

ASSUMPTION 1. $\mathcal{F}((F_k)_{k \in \llbracket K \rrbracket}) \neq \emptyset$.

A necessary and sufficient condition for satisfying Assumption 1 is to ensure that $\mathcal{P}_k \neq \emptyset$ for all $k \in \llbracket K \rrbracket$.

A vehicle that travels along the edge $e \in \mathcal{E}$ observes a cost (e.g., latency) of $\tilde{\ell}_e(\phi_e)$ with a given mapping $\tilde{\ell}_e : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$. Hence, a vehicle that uses the path $p \in \mathcal{P}$ observes a total cost of $\ell_p(f) = \sum_{e \in p} \tilde{\ell}_e(\phi_e)$. Note that we use $\tilde{\ell}_e$ and ℓ_p to, respectively, denote the cost of using edge $e \in \mathcal{E}$ and path $p \in \mathcal{P}$.

In this formulation, each player is an infinitesimal amount of flow that minimizes its cost by selecting its path. Now, we define the equilibrium for the introduced routing game.

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[2]We use the aggregate vector of edge flows $\phi = (\phi_e)_{e \in \mathcal{E}}$ and the aggregate vector of path flows $f = (f_p)_{p \in \mathcal{P}}$ interchangeably as there is a one-to-one correspondence between them.

DEFINITION 1. (WARDROP EQUILIBRIUM): *A flow vector $f = (f_p)_{p \in \mathcal{P}}$ is a Wardrop equilibrium for the routing game if, for all $k \in [\![K]\!]$, $f_p > 0$ for a path $p \in \mathcal{P}_k$ implies that $\ell_p(f) \leq \ell_{p'}(f)$ for all $p' \in \mathcal{P}_k$.*

This definition implies that for each source–destination pair (s_k, d_k), $k \in [\![K]\!]$, all the paths with a nonzero flow (i.e., the utilized paths) have equal latencies and the rest (i.e., the paths with a zero flow) have a larger (or equal) latency. Throughout this paper, we make the following assumption.

ASSUMPTION 2. *For all $e \in \mathcal{E}$, $\tilde{\ell}_e(\cdot)$ is (i) twice continuously differentiable, (ii) convex, and (iii) nondecreasing.*

This assumption guarantees that the problem of finding a Nash equilibrium boils down to solving a convex optimization.

We can now define the social cost function

$$C(f) = \sum_{p \in \mathcal{P}} f_p \ell_p(f) = \sum_{e \in \mathcal{E}} \phi_e \tilde{\ell}_e(\phi_e),$$

where the second equality can be proved following simple algebraic manipulations [21].

DEFINITION 2. (SOCIALLY OPTIMAL FLOW)*: A flow vector $f = (f_p)_{p \in \mathcal{P}}$ is a socially optimal flow for the routing game if $f \in \arg\min_{f' \in \mathcal{F}} C(f')$.*

The Wardrop equilibria can be inefficient (i.e., the social cost of the Wardrop equilibrium is larger than the social cost of a socially optimal flow) [22]. In the remainder of this section, we discuss imposing tolls on the edges of the graph $\mathcal{G}$ to reduce this inefficiency.

Let us assume that a driver must pay a toll $\tilde{\tau}_e(\phi_e)$, with $\tilde{\tau}_e : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, for using the edge $e \in \mathcal{E}$, where (as stated earlier) $\phi_e = \sum_{p \in \mathcal{P}: e \in p} f_p$ is the flow on edge $e \in \mathcal{E}$. Therefore, a vehicle that is using path $p \in \mathcal{P}_k$ endures a total cost of $\ell_p(f) + \tau_p(f)$, where $\tau_p(f)$ is total amount of money that this vehicle must pay for using path p and can be calculated as $\tau_p(f) = \sum_{e \in p} \tilde{\tau}_e(\phi_e)$. Hence, the definition of the equilibrium should be slightly modified.

DEFINITION 3. (WARDROP EQUILIBRIUM WITH TOLLS)*: A flow vector $f = (f_p)_{p \in \mathcal{P}}$ is a Wardrop equilibrium for the routing game with tolls if, for all $k \in [\![K]\!]$, $f_p > 0$ for a path $p \in \mathcal{P}_k$ implies that $\ell_p(f) + \tau_p(f) \leq \ell_{p'}(f) + \tau_{p'}(f)$ for all $p' \in \mathcal{P}_k$.*

In [19], Pigou suggested marginal cost taxes $\tilde{\tau}_e(\phi_e) = \phi_e(\mathrm{d}\tilde{\ell}_e(\phi_e)/\mathrm{d}\phi_e)$ as a way for reducing the inherent inefficiency of the equilibria in routing games. Later, it was proved that with using these tolls, the socially optimal flow becomes a Wardrop equilibria of the routing game with tolls [2]. These tolls are called marginal cost taxes since they correspond to the marginal increase in cost caused by adding one user to the edge $\mathrm{d}\tilde{\ell}_e(\phi_e)/\mathrm{d}\phi_e$ multiplied by the amount of the traffic that suffers from this increase ϕ_e. Although extremely effective, it is difficult to implement these taxes since they are flow dependent (i.e., the drivers do not know the actual value of tolls prior to using the road and, hence, they might not be able to make an informed decision). Hence, it is advantageous to devise an online method for setting the tolls adaptively to recover a socially optimal flow; however, we would like an scheme that results in piecewise-constant taxes over relatively large periods of time. This way, we can guarantee that the drivers have enough time to (re)calculate their preferred routes and make informed decisions.

3 Repeated Routing Game

Here, we assume that the routing game is played repeatedly on each day $n \in \mathbb{N}$ for an infinite horizon. On each day, the vehicles select their preferred path, which generate flows $f[n] = (f_p[n])_{p \in \mathcal{P}}$. Then, they observe the cost associated with each path, i.e., the actual travel cost and the imposed tolls, and use this information, accompanied with their (finite) memory, to select their path on the subsequent day(s). We consider the multiplicative update rule, which is a no-regret learning strategy (see [15] for more information on no-regret strategies) [10, 18]. In this strategy, the agents select their actions with a probability distribution inversely proportional to the exponential of the average cost that they have observed so far. Considering that there is a continuum of players in a repeated routing game, this amounts to dividing the players into various paths with portions inversely proportional to the exponential of the average cost of the paths. This results in Algorithm 1. Notice that, in the multiplicative update rule in Algorithm 1, for each driver to calculate the evolution of the weights $(w_p[n])_{p \in \mathcal{P}}$, she needs to have access the measurements of the costs (not necessarily the actual cost functions) at each iteration over all the available paths. Evidently, the driver can directly measure the cost of the path that she has chosen in that iteration. We assume the drivers can measure the costs of the alternative paths as well. This can be achieved by broadcasting the measurements from a central node (e.g., a traffic forecast unit in a radio station). In support of this assumption, experimental studies show that players sometimes notice and consider the costs and payoffs of the actions that they have not selected [7], even if they are not directly fed the information. An alternative approach could be to assume that drivers only update the

average cost of the paths that they have travelled. This may, however, result in asymmetric update rules that are more difficult to analyze and slower to converge.

Throughout the rest of the paper, we make the following assumption regarding the parameters of Algorithm 1.

ASSUMPTION 3. *Parameters $\rho_1, \rho_2 \in \mathbb{R}_{>0}$ are selected so that $\ell_p(f[n]) \leq \rho_1$ and $\tau_p[n] \leq \rho_2$ for all $p \in \mathcal{P}$ and all $n \in \mathbb{N}$.*

In the reminder of this section, we prove that the multiplicative update rule in Algorithm 1 converges to a socially optimal flow. To do so, first, we prove the following lemma.

LEMMA 3.1. *For Algorithm 1, we have*

$$
\begin{aligned}
f_p[n+1] =& f_p[n] + \frac{\epsilon[n]}{\rho_1 + \rho_2} f_p[n] \\
& \times \left[\left(\sum_{p' \in \mathcal{P}_k} \frac{f_{p'}[n]}{F_k} \left(\ell_{p'}(f[n]) + \tau_{p'}[n] \right) \right) \right. \\
& \left. - \left(\ell_p(f[n]) + \tau_p[n] \right) \right] + \mathcal{O}(\epsilon[n]^2).
\end{aligned}
$$

Proof. Due to space limitations, the proof is omitted. See [8] for the details. ∎

To present the rest of the results, let us, for each $p \in \mathcal{P}$, define the mapping

$$
\begin{aligned}
\eta_p :& \mathbb{R}_{\geq 0}^{|\mathcal{P}|} \to \mathbb{R} \\
& f \quad \mapsto \sum_{e \in p} \left[\phi_e \frac{\mathrm{d}\tilde{\ell}_e(\phi_e)}{\mathrm{d}\phi_e} \right]_{\phi_e = \sum_{p' \in \mathcal{P}: e \in p'} f_{p'}} .
\end{aligned}
$$

Evidently, the imposed piecewise-constant congestion taxes in Algorithm 1 can now be calculated as $\tau_p[n] = \eta_p(f[n - D_n])$, where $D_n = n - D\lfloor n/D \rfloor$.

LEMMA 3.2. *Let us select step sizes $\{\epsilon[n]\}_{n \in \mathbb{N}}$ either as*

- *$\epsilon[n] = \alpha/(n + \beta)$ for some $\alpha, \beta \in \mathbb{R}_{>0}$,*

- *$\epsilon[n] = \epsilon \in \mathbb{R}_{>0}$,*

for all $n \in \mathbb{N}$. Then, for Algorithm 1, we have

$$
\begin{aligned}
f_p[n+1] =& f_p[n] + \frac{\epsilon[n]}{\rho_1 + \rho_2} f_p[n] \\
& \times \left[\left(\sum_{p' \in \mathcal{P}_k} \frac{f_{p'}[n]}{F_k} \left(\ell_{p'}(f[n]) + \eta_{p'}(f[n]) \right) \right) \right. \\
& \left. - \left(\ell_p(f[n]) + \eta_p(f[n]) \right) \right] + \mathcal{O}(\epsilon[n]^2).
\end{aligned}
$$

Proof. Due to space limitations, the proof is omitted. See [8] for the details. ∎

This lemma shows that the time-varying delay for setting the tolls is not important so long as it is bounded (i.e., the delay does not change the update rule significantly).

THEOREM 3.1. *Let us select $\epsilon[n] = \alpha/(n + \beta)$ for some $\alpha, \beta \in \mathbb{R}_{>0}$ and for all $n \in \mathbb{N}$, and define $\mathcal{S} = \arg\min_{f \in \mathcal{F}} C(f)$. Then, for Algorithm 1, we get $\lim_{n \to \infty} \mathrm{dist}(\mathcal{S}, (f_p[n])_{p \in \mathcal{P}}) = 0$.*

Proof. Let us define the sequence $\{t[n]\}_{n \in \mathbb{N}}$ such that $t[0] = 0$ and $t[n + 1] - t[n] = \epsilon[n]$ for all $n \in \mathbb{N}$. For all $t \in \mathbb{R}_{\geq 0}$, we may define

$$
\bar{f}_p(t) = f_p[n] + (f_p[n + 1] - f_p[n]) \frac{t - t[n]}{t[n + 1] - t[n]}.
$$

This is a first-order interpolation of the discrete-time flow sequence $(f_p[n])_{n \in \mathbb{N}}$ for all $p \in \mathcal{P}$. Moreover, for all $t \in \mathbb{R}_{\geq \tau}$, we may define $\tilde{f}_p^\tau(t)$ as the unique solution of the ordinary differential equation in (3.1), where $\tilde{f}^\tau(t) = (\tilde{f}_p^\tau(t))_{p \in \mathcal{P}}$ and $\bar{f}(t) = (\bar{f}_p(t))_{p \in \mathcal{P}}$. We can prove that the mapping on the right-hand side of (3.1) is Lipschitz continuous (see [8] for the details). Now, combining the results of Lemma 3.2 in this paper and Lemma 1 in [6, Ch. 2, p. 12], specifically, from the third extension introduced in Section 2.2 of [6, Ch. 2, p. 17], we can see that $\lim_{\tau \to \infty} \lim_{t \in [\tau, \tau + T]} \|\tilde{f}^\tau(t) - \bar{f}(t)\| = 0, \forall T \in \mathbb{R}_{>0}$. Evidently, the set of socially optimal solutions $\mathcal{S}$ is an invariant set of the ordinary differential equations in (3.1). This holds because if $\tilde{f}_p^\tau(t) \in \mathcal{S}$, $\forall p \in \mathcal{P}$, we get

$$
\begin{aligned}
& \ell_p(\tilde{f}^\tau(t)) + \eta_p(\tilde{f}^\tau(t)) \\
& \qquad = \sum_{p' \in \mathcal{P}_k} \frac{\tilde{f}_{p'}^\tau(t)}{F_k} \left(\ell_{p'}(\tilde{f}^\tau(t)) + \eta_{p'}(\tilde{f}^\tau(t)) \right).
\end{aligned}
$$

To show the next step, first, we should prove that $\tilde{f}_p^\tau(t)/F_k \geq 0$ and $\sum_{p \in \mathcal{P}_k} \tilde{f}_p^\tau(t)/F_k = 1$ for all $k \in [\![K]\!]$ and $p \in \mathcal{P}_k$. The first property that $\tilde{f}_p^\tau(t)/F_k \geq 0$ follows directly from the ordinary differential equation in (3.1). For the second property note that by definition of the initial point, we have $\sum_{p \in \mathcal{P}_k} \tilde{f}_p^\tau(\tau)/F_k = 1$; see Algorithms 1 in conjunction with the definition of the interpolation for constructing $\bar{f}(t)$. Now, we have

$$
\frac{\mathrm{d}}{\mathrm{d}t} \left[\sum_{p \in \mathcal{P}_k} \frac{1}{F_k} \tilde{f}_p^\tau(t) \right] = \sum_{p \in \mathcal{P}_k} \frac{1}{F_k} \frac{\mathrm{d}}{\mathrm{d}t} \tilde{f}_p^\tau(t) = 0, \forall t \in \mathbb{R}_{\geq \tau},
$$

and, as a result, $\sum_{p \in \mathcal{P}_k} \tilde{f}_p^\tau(t)/F_k = \sum_{p \in \mathcal{P}_k} \tilde{f}_p^\tau(\tau)/F_k = 1$ for all $t \in \mathbb{R}_{\geq \tau}$. Using this property of flows,

Algorithm 1 Multiplicative update rule with piecewise-constant congestion taxing policy.

Input: $\{\epsilon[n]\}_{n\in\mathbb{N}}$ and $\rho_1, \rho_2 \in \mathbb{R}_{>0}$
1: Initialize $w_p[1] = 1, \forall p \in \mathcal{P}_k, \forall k \in [\![K]\!]$
2: Initialize $\tilde{\tau}_e[n'] = 0, \forall e \in \mathcal{E}, \forall n' \in [\![D]\!]$
3: **for** $n = 1, 2, \cdots$ **do**
4: Calculate the flows $f_p[n] = F_k w_p[n]/(\sum_{p'\in\mathcal{P}_k} w_{p'}[n]), \forall p \in \mathcal{P}_k, \forall k \in [\![K]\!]$
5: Update the weights $w_p[n+1] = w_p[n] \exp(-\epsilon[n]/(\rho_1 + \rho_2) \sum_{e\in p}(\tilde{\ell}_e(f[n]) + \tilde{\tau}_e[n]))$
6: **if** $n \equiv 0 \,(\mathrm{mod}\, D)$ **then**
7:   Set the tolls for the next D days $\tilde{\tau}_e[n'] = \left[\phi_e \frac{\mathrm{d}\tilde{\ell}_e(\phi_e)}{\mathrm{d}\phi_e}\right]_{\phi_e = \sum_{p\in\mathcal{P}:e\in p} f_p[n]}, \; \forall e \in \mathcal{E}, \forall n' \in \mathbb{N}: n' - n \in [\![D]\!]$
8: **end if**
9: **end for**

(3.1)

$$\frac{\mathrm{d}}{\mathrm{d}t}\tilde{f}_p^\tau(t) = \frac{1}{\rho_1+\rho_2}\tilde{f}_p^\tau(t)\left[\left(\sum_{p'\in\mathcal{P}_k}\frac{\tilde{f}_{p'}^\tau(t)}{F_k}\left(\ell_{p'}(\tilde{f}^\tau(t)) + \eta_{p'}(\tilde{f}^\tau(t))\right)\right) - \left(\ell_p(\tilde{f}^\tau(t)) + \eta_p(\tilde{f}^\tau(t))\right)\right], \; \tilde{f}^\tau(\tau) = \bar{f}(\tau),$$

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}C(\tilde{f}_p^\tau(t)) &= \sum_{p\in\mathcal{P}}\left.\frac{\partial C(f)}{\partial f_p}\right|_{f=(\tilde{f}_p^\tau(t))_{p\in\mathcal{P}}}\frac{\mathrm{d}}{\mathrm{d}t}\tilde{f}_p^\tau(t) \\
&= \frac{1}{\rho_1+\rho_2}\sum_{k\in[\![K]\!]}\sum_{p\in\mathcal{P}_k}\left(\ell_p(\tilde{f}^\tau(t)) + \eta_p(\tilde{f}^\tau(t))\right)\tilde{f}_p^\tau(t) \\
&\quad\times\left[\left(\sum_{p'\in\mathcal{P}_k}\frac{\tilde{f}_{p'}^\tau(t)}{F_k}\left(\ell_{p'}(\tilde{f}^\tau(t)) + \eta_{p'}(\tilde{f}^\tau(t))\right)\right) - \left(\ell_p(\tilde{f}^\tau(t)) + \eta_p(\tilde{f}^\tau(t))\right)\right] \\
&= \frac{1}{\rho_1+\rho_2}\sum_{k\in[\![K]\!]}F_k\left[\left(\sum_{p'\in\mathcal{P}_k}\frac{\tilde{f}_{p'}^\tau(t)}{F_k}\left(\ell_{p'}(\tilde{f}^\tau(t)) + \eta_{p'}(\tilde{f}^\tau(t))\right)\right)^2 - \sum_{p\in\mathcal{P}}\frac{\tilde{f}_p^\tau(t)}{F_k}\left(\ell_p(\tilde{f}^\tau(t)) + \eta_p(\tilde{f}^\tau(t))\right)^2\right]
\end{aligned}$$

(3.2) $\qquad \leq 0,$

we can prove (3.2), where the last inequality follows from Jensen's inequality (when using the fact that the mapping $x \mapsto x^2$ is a convex function). Because the mapping $x \mapsto x^2$ is strictly convex, the equality in (3.2) holds if and only if $\ell_{p''}(\tilde{f}^\tau(t)) + \eta_{p''}(\tilde{f}^\tau(t)) = \ell_{p'}(\tilde{f}^\tau(t)) + \eta_{p'}(\tilde{f}^\tau(t))$ for any two $p', p'' \in \mathcal{P}$ such that $\tilde{f}_{p'}^\tau(t), \tilde{f}_{p''}^\tau(t) \neq 0$. This is the definition of $\mathcal{S}$. Therefore, the equality in (3.2) holds if and only if $\tilde{f}^\tau(t) \in \mathcal{S}$. By defining the Lyapunov function in Corollary 3 in [6, Ch. 2, p. 15] as $V(f) = C(f) - \min_{f\in\mathcal{F}} C(f)$, we can see that $\{(f_p[n])_{p\in\mathcal{P}}\}_{n\in\mathbb{N}}$ converges to an internally chain transitive invariant set contained in $\mathcal{S}$. ∎

Unfortunately, the shrinking step sizes in Theorem 3.1 renders the algorithms impractical for the cases where the parameters of the routing game (e.g., the demands over the source–destination nodes) are time varying since the algorithm cannot adapt itself fast enough (especially after many steps because the step size is very small). This observation motivates using a constant step size, however, the price for such a selection is that we can only converge to a neighborhood of the socially optimal flow.

THEOREM 3.2. *Let us select $\epsilon[n] = \epsilon \in \mathbb{R}_{>0}$ for all $n \in \mathbb{N}$ and define $\mathcal{S} = \arg\min_{f\in\mathcal{F}} C(f)$. Then, for Algorithm 1, we get $\lim_{n\to\infty} \mathrm{dist}(\mathcal{S}, (f_p[n])_{p\in\mathcal{P}}) = \mathcal{O}(\epsilon)$.*

Proof. The proof follows the same line of reasoning as in the proof of Theorem 3.1; however, it builds upon using Lemma 1 and Theorem 3 of [6, Ch. 9, pp. 103–114]. ∎

So long as the step size ϵ is large enough the algorithm can adapt rapidly to the changes in the parameters of the routing game; however, the solution can potentially be far from the socially optimal flow. By reducing the step size, we can achieve a better solution (in terms of the social cost function) but the algorithm, in such case, would respond slower to the changes.
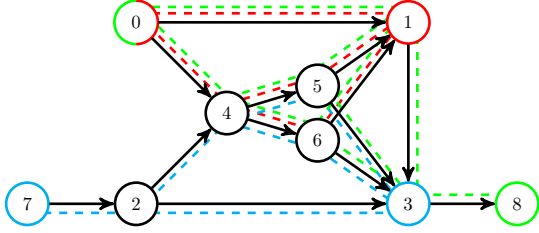
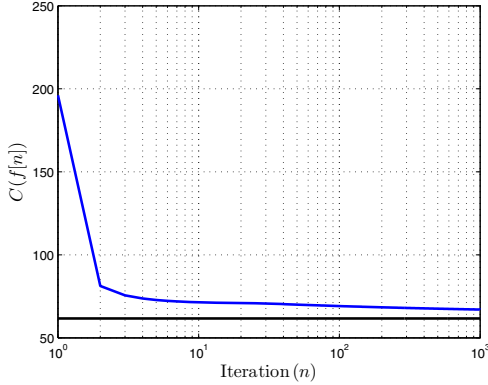Figure 2: An example of a transportation network.

Figure 3: The social cost $C(f[n])$ versus the iteration number n for the example with constant demand.
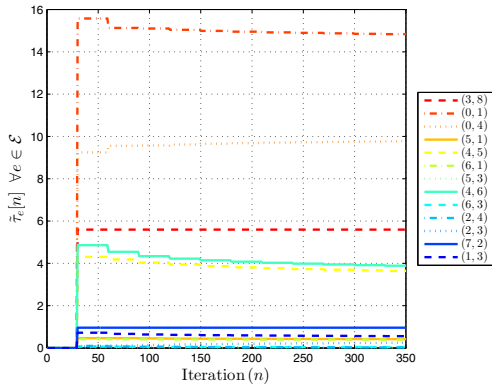

Figure 4: The congestion taxes $(\tilde{\tau}_e[n])_{e \in \mathcal{E}}$ versus the iteration number n for the example with constant demand.

4 Numerical Example

Consider the transportation network portrayed by the directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ in Figure 2. We assume that there are $K = 3$ source–destination nodes $(s_1, t_1) = (0, 1)$, $(s_2, t_2) = (7, 3)$, and $(s_3, t_3) = (0, 8)$. In Figure 2, each source–destination pair and its corresponding paths are portrayed in a separate color. We

adopt a widely used model for the edge cost functions, namely, the Bureau of Public Roads model for the delay [26], which is given by $\tilde{\ell}_e(\phi_e) = (d_e/v_e^{\max})[1 + 0.15\,(\phi_e/c_e)^4], \forall e \in \mathcal{E}$, where $d_e \in \mathbb{R}_{\geq 0}$ is the length of the road, $v_e^{\max} \in \mathbb{R}_{>0}$ is the speed limit, and c_e is the capacity of the road (e.g., approximately 2000 vehicle/h multiplied by the number of lanes [20]). In this cost function, for all $e \in \mathcal{E}$, we set the speed limit as $v_e^{\max} = 70\,\mathrm{km/h}$ and set the capacity of the road $c_e = 2000\,\mathrm{vehicle/h}$ (as recommended for single lane roads). Tthe length of each road d_e, $e \in \mathcal{E}$, is presented in Table 1.

4.1 Fixed Demand First, we consider the case where the total flows that need to pass through source–destination nodes are constant and equal to $F_1 = 8000\,\mathrm{vehicle/h}$, $F_2 = 3000\,\mathrm{vehicle/h}$, and $F_3 = 4000\,\mathrm{vehicle/h}$. We set $D = 30$, which means that the congestion taxes get updated monthly. Finally, let us use vanishing step sizes $\epsilon[n] = 1/n$ for all $n \in \mathbb{N}$. Figure 3 illustrates the social cost of the flows extracted from Algorithm 1 as a function of the iteration numbers (solid blue curve) as well as the cost of the socially optimal flow (solid black curve). As we expect, the social cost of the extracted flows approaches the cost of the socially optimal flow. Figure 4 illustrates the congestion charges for various edges in the transportation network $\tilde{\tau}_e[n]$, $e \in \mathcal{E}$, versus the iteration number n. As we expect, the drivers on highly congested roads, e.g., $(0, 1)$, should pay much more to be persuaded to use less-congested alternatives (that are perhaps longer or less convenient for them). Figure 5 (left) portray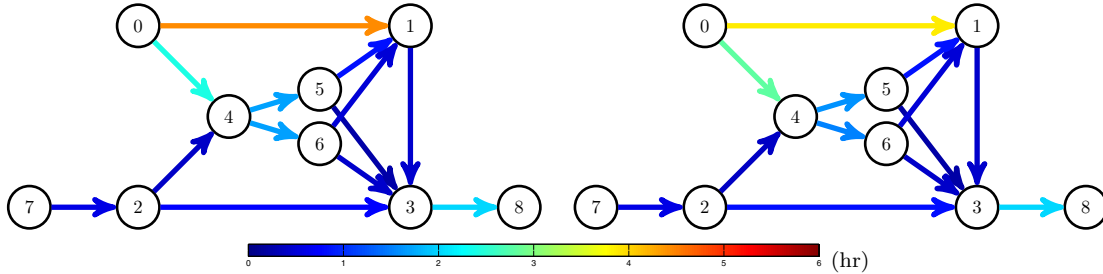s the delays over the roads in the transportation network at the Wardrop equilibrium of the routing game *in the absence* of congestion taxes. In contrast, Figure 5 (right) illustrates the delays over the roads at the Wardrop equilibrium of the routing game *in the presence* of congestion taxes. As we expect, with imposing taxes, a portion of the flow (i.e., some of the vehicles) switch from highly congested roads, e.g., $(0, 1) \in \mathcal{E}$, to slightly less congested roads, e.g., $(0, 4) \in \mathcal{E}$, at the expense of taking a longer path which is now desirable because of the high level of congestion taxes over the shorter path. This behaviour improves the social cost function by 4.6% in this example.

4.2 Time-Varying Demand Now, consider the case where the total flows for various source–destination nodes vary with time as in Figure 6. In this case, we use Algorithm 1 with a constant step size $\epsilon[n] = 5 \times 10^{-2}$ for all $n \in \mathbb{N}$. Figure 7 illustrates the congestion taxes versus the iterations of the algorithm. Clearly, the algorithm updates these congestion taxes in response to the

Table 1: Length of roads in the transportation network employed for the numerical example in Section 4.

e	$(3,8)$	$(0,1)$	$(0,4)$	$(5,1)$	$(4,5)$	$(6,1)$	$(5,3)$	$(4,6)$	$(6,3)$	$(2,4)$	$(2,3)$	$(7,2)$	$(1,3)$
$d_e(\mathrm{km})$	40.81	17.22	17.68	52.77	55.15	29.16	12.19	45.14	26.16	25.13	46.47	22.1	25.22

Figure 5: The delays over the roads in the transportation network at the Wardrop equilibrium of the routing game in the absence of congestion taxes (left) and in the presence of congestion taxes (right).
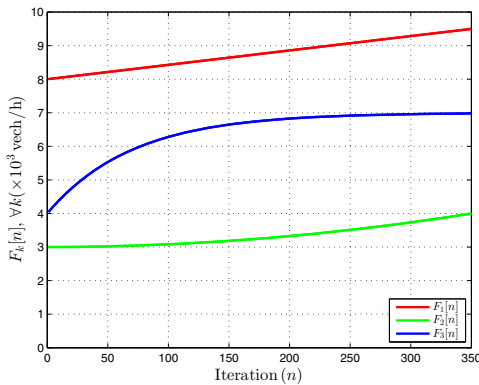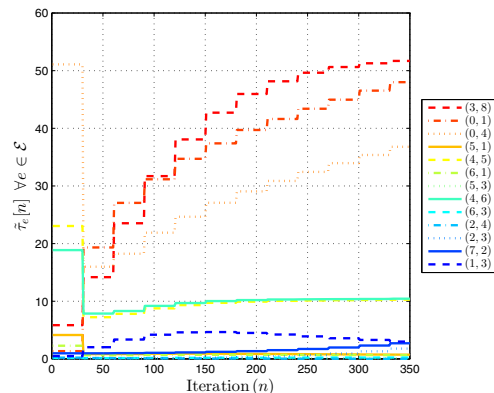

Figure 6: Time-varying demand.

Figure 7: The congestion taxes for the numerical example with time-varying demand.

changes in the demand. Now, allow us to define $f^*[n]$ to be the socially optimal flow for demands $(F_k[n])_{k=1}^3$ in Figure 6. Figure 8 illustrates $C(f[k])/C(f^*[n]) - 1$ as a function of the iteration numbers. Evidently, the smaller $C(f[k])/C(f^*[n]) - 1$ is, the closer the social cost of the generated flow is to the cost of the socially optimal flow. This figure clearly show that the algorithm closely follows the socially optimal flow.

5 Conclusions

We studied repeated routing game with piecewise-constant congestion taxing policy. We used the multiplicative update rule with both vanishing and constant step sizes. For vanishing step sizes, we proved the convergence to the set of socially optimal flows; however, using constant step sizes, we could only prove the convergence to a neighbourhood of the socially optimal flows. Future research can focus on devising piecewise-constant congestion charges policies for only a subset of
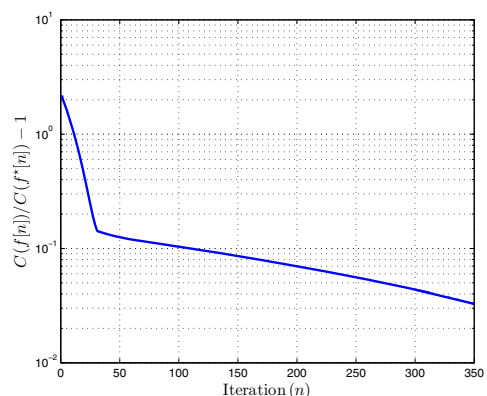

Figure 8: A measure of the efficiency of the extracted flows $f[n]$ versus the iteration number n for the numerical example with time-varying demand.

the edges in the transportation network. We can also focus on the multi-class traffic to understand the influence of the drivers' value-of-time.

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