IT for Statistics and Learning
2023

Assignment 1
Assigned: Friday, Nov 10, 2023
Due: Friday, Nov 17, 2023 M. Skoglund

Problem 1.1: Let $X \geq 0$ be an integer-valued (discrete) random variable with $p(x) = \Pr(X = x)$, $x = 0, 1, 2, \ldots$. Specify the pmf $p(x)$ that maximizes $H(X)$ subject to the constraint $E[X] = m > 0$.

Problem 1.2: Assume $\{X_n\}$ is a stationary discrete-valued random sequence. Prove that

$$\frac{1}{n} H(X_1, \ldots, X_n)$$

is non-increasing in n

Problem 1.3: Consider three random variables X, Y and Z, each with values in $\{1, \ldots, M\}$, $M < \infty$. Assume that every pair is pairwise independent, i.e. $I(X;Y) = I(X;Z) = I(Z;Y) = 0$; and that each marginal is uniform, i.e. $H(X) = H(Y) = H(Z) = \log M$. Give a tight lower bound on $H(X, Y, Z)$, and describe (e.g. in the case $M = 2$) a joint distribution that achieves it.

Problem 1.4: Prove that $D(P\|Q)$ is convex in (P, Q), that is

$$D(\lambda P_1 + (1-\lambda)P_2 \| \lambda Q_1 + (1-\lambda)Q_2) \leq \lambda D(P_1\|Q_1) + (1-\lambda)D(P_2\|Q_2)$$

for $\lambda \in [0, 1]$

Problem 1.5: Given P and Q on $(\Omega, \mathcal{A})$, prove that

$$\frac{1}{2} E_Q \left| \frac{dP}{dQ} - 1 \right| = \sup_{A \in \mathcal{A}} (P(A) - Q(A))$$

Problem 1.6: Given P and Q on $(\Omega, \mathcal{A})$, prove that

$$\frac{1}{2} E_Q \left| \frac{dP}{dQ} - 1 \right| = \frac{1}{2} \sum_x |p(x) - q(x)|$$

if P and Q have pmfs p and q (the discrete case)

Problem 1.7: Prove that $D(P\|Q) \leq \log(1 + \chi^2(P\|Q))$