IT for Statistics and Learning
2024

Assignment 13
Assigned: Thu, Feb 15, 2024
Due: before the lecture on Fr, Feb 23, 2024 — T. Oechtering

Problem 13.1: *Complete the proof.* Show that the Neyman-Pearson optimal test $g(x^n) = \begin{cases} H_0, & \text{if } x^n \in \mathcal{D}, \\ H_1, & \text{if } x^n \notin \mathcal{D} \end{cases}$ with decision region $\mathcal{D}_n(T) = \left\{ x^n \in \mathcal{A}^n \,:\, \frac{P_0^n(x^n)}{P_1^n(x^n)} > T \right\}$ can be equivalently written as $D(\hat{P}_{x^n}||P_1) - D(\hat{P}_{x^n}||P_0) \underset{H_1}{\overset{H_0}{\gtrless}} \frac{1}{n} \log T$

Problem 13.2: *Hypothesis testing.* Let $X_i \overset{iid}{\sim} P$ defined on $\mathcal{A} = \{0, 1, 2, 3\}$. Consider two hypothesis $H_0 : P(x) = P_0(x)$ and $H_1 : P(x) = P_1(x)$ with

$$P_0(x) = \begin{cases} \frac{1}{8}, & \text{if } x = 0 \\ \frac{1}{2}, & \text{if } x = 1 \\ \frac{1}{8}, & \text{if } x = 2 \\ \frac{1}{4}, & \text{if } x = 3 \end{cases} \qquad P_1(x) = \begin{cases} \frac{1}{4}, & \text{if } x = 0 \\ \frac{1}{8}, & \text{if } x = 1 \\ \frac{1}{2}, & \text{if } x = 2 \\ \frac{1}{8}, & \text{if } x = 3 \end{cases}$$

Find the error exponent for Prob$\{$decide $H_1|H_0$ true$\}$ in the best hypothesis test subject to Prob$\{$decide $H_0|H_1$ true$\} \leq \frac{1}{2}$ and Prob$\{$decide $H_0|H_1$ true$\} \leq \frac{1}{4}$.

Problem 13.3: *Conditional limiting distribution.* Find the exact value of Prob$\left\{X_1 = 1 \middle| \frac{1}{n} \sum_{i=1}^{n} X_i = \frac{1}{4}\right\}$ if $X_1, X_2, \ldots, X_n$ are Bernoulli($\frac{2}{3}$) and n is a multiple of 4.

Problem 13.4: *Variational inequality.* Verify for positive random variables X that

$$\log E_P(X) = \sup_Q \left[E_Q(\log X) - D(Q||P) \right]$$

with expectation $E_P(X) = \sum_x P(x)$ and relative entropy $D(Q||P) = \sum_x Q(x) \log \frac{Q(x)}{P(x)}$ and the supremum is over all $Q(x) \geq 0$, $\sum_x Q(x) = 1$.

Hint: It is enough to extremize $J(Q) = E_Q \ln X - D(Q||P) + \lambda(\sum_x Q(x) - 1)$