IT for Statistics and Learning
2023

Assignment 2

Assigned: Friday, Nov 17, 2023
Due: Thursday, Nov 23, 2023 M. Skoglund

Problem 2.1: Consider $X \in \{0,1\}$ with $p = \Pr(X = 1) \leq 1/2$. We want to represent X as $Y \in \{0,1\}$ through a random transformation $P_{Y|X}$ with distortion characterized via

$$d(x,y) = \begin{cases} 0 & y = x \\ 1 & y \neq x \end{cases}$$

(Hamming distortion). Find $R(D)$ and $D(R)$ for this scenario.

Problem 2.2: Let W be zero-mean Gaussian with variance $E[W^2] = 1$, and let X be absolutely continuous and independent of W. Consider

$$Y_s = \sqrt{s}X + W$$

with $0 < s < \infty$ and assuming $E[X^2] < \infty$. Let $g^*(y)$ be the (Borel measurable) function that minimizes the MSE $E[(X - g(Y_s))^2]$. It can then be proved that

$$\frac{d}{ds} I(X;Y_s) = \frac{1}{2} E[(X - g^*(Y_s))^2]$$

(with the mutual information $I(X;Y_s)$ in nats). Verify this result for the special case that X is zero-mean Gaussian.

Problem 2.3: Prove the n-dimensional bound

$$|E[(X - \hat{x}(Y))^T (X - \hat{x}(Y))]| \geq \frac{1}{(2\pi e)^n} 2^{2h(X|Y)}$$

Problem 2.4: Let X be exponentially distributed with variance $\sigma^2/2$, pdf $f(x)$ and rate–distortion function $R(D)$ (for $d(x,y) = (x-y)^2$). Verify that

$$\lim_{D \to 0} \frac{2R(D)}{\log \frac{\sigma^2}{D} - 2D(f(x)\|g(x))} = 1$$

for $g(x) =$ zero-mean Gaussian with variance σ^2

Problem 2.5: As discussed in Lec. 2, an iterative approach to computing $R(D)$ can be based on minimizing

$$\int \left\{ \int \log \frac{dP_{Y|X=x}}{dQ_Y} dP_{Y|X=x} \right\} dP_X + \lambda E[d(X,Y)]$$

over $P_{Y|X=x}$ for fixed Q_Y and then over Q_Y for (the updated) fixed $P_{Y|X=x}$, and so on. In the case that all random variables are *discrete*, i.e. with pmf's, derive explicit necessary criteria in terms of λ and these pmf's.