Probability and Random Processes
2024

Assignment 3
Assigned: Thursday, February 1, 2024
Due: Friday, February 9, 2024 M. Skoglund

Problem 3.1: Define/motivate the concepts σ-algebra, measure, measure space, measurable space and measurable set.

Problem 3.2: On the real line, define/motivate Borel measurable set and Borel measurable function. Discuss the relation between Borel measurable and continuous functions.

Problem 3.3: Given a general measure space, describe what it means for a real-valued function to be measurable. Given two general measure spaces, define the concept of measurable transformation between the spaces.

Problem 3.4: Define and discuss the concepts almost everywhere (a.e.), convergence a.e., and convergence in measure. For a finite measure, prove that convergence a.e. implies convergence in measure (you can use the result in Problem 3.5 without proof). Describe a counterexample showing that the reverse statement does not hold in general.

Problem 3.5: Given a measure space $(\Omega, \mathcal{A}, \mu)$, let $\{E_n\}$ be an infinite sequence of $\mathcal{A}$-measurable sets, with $\mu(\cup_n E_n) < \infty$. Prove that

$$\mu\left(\bigcup_{n=1}^{\infty}\left(\bigcap_{k=n}^{\infty} E_k\right)\right) \leq \liminf_{n\to\infty} \mu(E_n) \leq \limsup_{n\to\infty} \mu(E_n) \leq \mu\left(\bigcap_{n=1}^{\infty}\left(\bigcup_{k=n}^{\infty} E_k\right)\right)$$

You can use the following facts without proof:

- For $A_i \in \mathcal{A}$ such that $A_1 \supset A_2 \supset \cdots$ and $\mu(A_1) < \infty$ it holds that

$$\mu\left(\bigcap_{i=1}^{\infty} A_i\right) = \lim_{i\to\infty} \mu(A_i)$$

- For $B_i \in \mathcal{A}$ such that $B_1 \subset B_2 \subset \cdots$ it holds that

$$\mu\left(\bigcup_{i=1}^{\infty} B_i\right) = \lim_{i\to\infty} \mu(B_i)$$