# Probability and Random Processes

## Lecture 0

- Course introduction
- Some basics



# Why This Course?

- Provide a first principles introduction to measure theory, probability and random processes
- Tailor the course to PhD students in information and signal theory, decision and control, and learning
- Why? — many very important results require that the reader knows at least the language/basics of measure theoretic probability

# Some Basics

- $\mathbb{R}$ = the real numbers
- $\mathbb{R}^* = \mathbb{R} \cup \{\infty, -\infty\}$ = the extended real numbers
- $\mathbb{Q}$ = the rational numbers
- $\mathbb{Z}$ = the integers
- $\mathbb{N}$ = the positive integers (natural numbers)



- A set $A$ of real numbers
  - $a = \sup A$ = least upper bound = smallest number $a$ such that $x \leq a$ for all $x \in A$
  - $b = \inf A$ = greatest lower bound = largest number $b$ such that $x \geq b$ for all $x \in A$
- Density of $\mathbb{Q}$ in $\mathbb{R}$
  - between any two real numbers, there is a rational number
  - between any two rational numbers, there is a real number

- A set $A \subset \mathbb{R}$ is open if for any $x \in A$ there is an $\varepsilon > 0$ such that $(x - \varepsilon, x + \varepsilon) \subset A$
  - $A \subset \mathbb{R}$ is an open set $\iff$ $A =$ countable union of disjoint open intervals
- The number $b$ is a limit point of the set $B$ if any open set (open interval) containing $b$ also contains a point from $B$
- The closure of $B = \{$ all $B$'s limit points $\}$
- $B$ is closed if it's equal to its closure $\iff$ $B^c$ is open



- A sequence $\{x_n\}$, $x_n \in \mathbb{R}$
- $a = \limsup x_n \iff$ for any $\varepsilon > 0$ there is an $N$ such that
  - $x_n < a + \varepsilon$ for all $n > N$
  - $x_n > a - \varepsilon$ for infinitely many $n > N$
- $b = \liminf x_n \iff$ for any $\varepsilon > 0$ there is an $N$ such that
  - $x_n > b - \varepsilon$ for all $n > N$
  - $x_n < b + \varepsilon$ for infinitely many $n > N$
- $c = \lim x_n \iff a = b = c$

- A function $f : \mathbb{R} \to \mathbb{R}$
  - $a = \lim_{x \to b} f(x) \iff$ for any $\varepsilon > 0$ there is a $\delta > 0$ such that $|f(x) - a| < \varepsilon$ for all $x \in (b - \delta, b + \delta) \setminus \{b\}$
- $f$ is continuous if $\lim_{x \to b} f(x) = f(b)$

$\iff$ $f^{-1}(A)$ open for each open $A \subset \mathbb{R}$, where

$$f^{-1}(A) = \{x : f(x) \in A\}$$



- A sequence of functions $\{f_n(x)\}$
  - $f_n \to f$ pointwise if $\{f_n(a)\}$ has a limit for any fixed number $a$, that is, for any $\varepsilon > 0$ there is an $N(a)$ such that $|f_n(a) - f(a)| < \varepsilon$ for all $n > N(a)$
  - $f_n \to f$ uniformly if for any $\varepsilon > 0$ there is an $N$ (that does not depend on $x$) such that $|f_n(x) - f(x)| < \varepsilon$ for all $n > N$ and for all $x$

- The set of continuous functions is closed under uniform but not under pointwise convergence
- If all the $f_n$'s in $\{f_n(x)\}$ are Riemann integrable, then $f = \lim f_n$ is Riemann integrable if the convergence is uniform, but not necessarily if it's pointwise
  - important part of the reason that we will need to look at the Lebesgue integral instead...