Probability and Random Processes

Lecture 11

- Measurable dynamical systems
- Random processes as dynamical systems
- Stationarity
- Ergodic theory

Measurable Dynamical System

- A probability space $(\Omega, \mathcal{A}, P)$
- A measurable transformation $\phi : (\Omega, \mathcal{A}) \to (\Omega, \mathcal{A})$
- The space $(\Omega, \mathcal{A}, P, \phi)$ is called a measurable dynamical system

Interpretation

- Nature selects an initial state $\omega = \omega_0$
- For $n \geq 0$, time acts on $\omega \in \Omega$ to 'move around' points in Ω,

$$\omega_{n+1} = \phi(\omega_n) = \{\,\text{notation}\,\} = \phi\omega_n = \phi^{n+1}\omega_0$$

  producing an orbit $\{\omega_n\}$
- The orbit is random because ω_0 is selected at random

Random Process as Dynamical System

- There are several ways to model a random process as a dynamical system – consider a discrete-time process $\{X_t\}$, for $t \in T$ and with $T = \mathbb{N}$
- Approach 1: $\{X_t\}$ is a collection of random variables $X_t : \Omega \to \mathbb{R}$ where for each t and ω

$$X_t(\omega) = X(\phi^t \omega)$$

  for some fixed random variable X
- Approach 2: Define ϕ implicitly on Ω by specifying a time-shift ϕ' on T, i.e., $\phi'(t) = t + 1$ and

$$(X_0, X_1, X_2, X_3, \ldots) \stackrel{\phi'}{\to} (X_1, X_2, X_3, X_4, \ldots)$$

- The model $X_t(\omega) = X(\phi^t \omega)$ fits with interpreting $\{X_t\}$ as a collection or random variables
- Note that we can also consider $(E^T, \mathcal{E}^T, \mu, \phi)$, i.e., the evolution of the process is described by $\phi : E^T \to E^T$
  - for example, $\phi(f(t)) = f(t+1)$

  which fits better with the time-shift in approach 2

Continuous Time

- We will focus on systems $(\Omega, \mathcal{A}, P, \phi)$ interpreting ϕ as 'one discrete action of time' – i.e., discrete-time systems
- Continuous-time systems can be modeled using a family $\{\phi_t\}_{t\in\Phi}$ of transformations, with $\Phi = \mathbb{R}$ or $\mathbb{R}^+$, such that

$$\phi_{t+s} = \phi_t \phi_s$$

  That is $\phi_{t+s}\omega = \phi_t \phi_s \omega$
- The family $\{\phi_t\}$ is called a flow
- A random waveform can be defined, e.g., as

$$X_t(\omega) = X(\phi_t \omega)$$

Stationarity

- A dynamical system $(\Omega, \mathcal{A}, P, \phi)$
- The system is stationary if

$$P(A) = P(\phi^{-1}(A)), \quad \text{for all } A \in \mathcal{A}$$

(where $\phi^{-1}(A) = \{\omega : \phi\omega \in A\} \subset \mathcal{A}$)

- The system is asymptotically mean stationary (AMS) if the limit

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} P(\phi^{-i}(A)) = \bar{P}(A)$$

exists pointwise for all $A \in \mathcal{A}$

- $(\Omega, \mathcal{A}, P, \phi)$ AMS $\Rightarrow$ $\bar{P}$ is a probability measure and $(\Omega, \mathcal{A}, \bar{P}, \phi)$ is stationary

(of course) stationary $\Rightarrow$ AMS and $\bar{P} = P$

Recurrence

- A dynamical system $(\Omega, \mathcal{A}, P, \phi)$
- A point $\omega \in A$ is said to be recurrent with respect to $A \in \mathcal{A}$ if there is a finite $N = N_A(\omega)$ such that $\phi^N \omega \in A$
  - $\omega \in A \Rightarrow \omega$ returns to A in finite time
- $A \in \mathcal{A}$ is recurrent if $P(A) > 0$ and $P(\{\omega \in A : \omega \text{ is not recurrent w.r.t. } A\}) = 0$
- $(\Omega, \mathcal{A}, P, \phi)$ is recurrent if all $A \in \mathcal{A}$ are recurrent
- stationary $\Rightarrow$ recurrent

Ergodicity

- A dynamical system $(\Omega, \mathcal{A}, P, \phi)$
- If $A \in \mathcal{A}$ is such that $\phi^{-1}(A) = A$ then A is invariant
- Let $\mathcal{I} = \{\text{all invariant } A \in \mathcal{A}\}$
  - $\mathcal{I}$ is a σ-algebra (why?)
- $(\Omega, \mathcal{A}, P, \phi)$ is ergodic if $P(B) = 1$ or 0 for all $B \in \mathcal{I}$

The Ergodic Theorem

- Theorem: If $(\Omega, \mathcal{A}, P, \phi)$ is AMS and X is a random variable such that
$$\int X(\omega) d\bar{P} < \infty$$
then
$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} X(\phi^i \omega) = \bar{E}[X|\mathcal{I}]$$
with probability one (under P and $\bar{P}$), where $\bar{E}[X|\mathcal{I}]$ is conditional expectation w.r.t. $\bar{P}$ and where $\mathcal{I}$ is the σ-algebra of invariant events

- Let

$$\bar{X} = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} X(\phi^n \omega)$$

then $\bar{X}$ is an $\mathcal{I}$-measurable random variable
- Note that

$$\int \bar{X} d\bar{P} = \int X d\bar{P}$$

and that if $(\Omega, \mathcal{A}, P, \phi)$, in addition, is ergodic then

$$\bar{X} = \int X d\bar{P}, \quad \bar{P}\text{-a.e.}$$

- In particular, if $(\Omega, \mathcal{A}, P, \phi)$ is stationary and ergodic, then

$\bar{X} = E[X]$ with probability one

Ergodic Decomposition

- Fix a standard measurable space $(\Omega, \mathcal{A})$ and $\phi : \Omega \to \Omega$
- Assume there is a P such that $(\Omega, \mathcal{A}, P, \phi)$ is stationary
- Then there are a standard space $(\Gamma, \mathcal{S})$, a family $\{P_\gamma\}_{\gamma\in\Gamma}$ and a measurable transformation $\tau : \Omega \to \Gamma$ such that
  1. τ is invariant ($\tau(\phi\omega) = \tau(\omega)$)
  2. $(\Omega, \mathcal{A}, P_\gamma, \phi)$ is stationary and ergodic, for each $\gamma \in \Gamma$
  3. if τ induces $P^*(S) = P(\tau^{-1}(S))$ on $(\Gamma, \mathcal{S})$, then for all $A \in \mathcal{A}$

$$P(A) = \int P_{\tau(\omega)}(A) dP(\omega) = \int P_\gamma(A) dP^*(\gamma)$$

  4. if $\int f(\omega) dP(\omega) < \infty$ then

$$\int f dP = \int \left\{ \int f dP_{\tau(\omega)} \right\} dP(\omega) = \int \left\{ \int f dP_\gamma \right\} dP^*(\gamma)$$

- For any stationary $(\Omega, \mathcal{A}, P, \phi)$ we can decompose P into a mixture of stationary and ergodic components P_γ
- For $A \in \mathcal{A}$, the component in effect is characterized by

$$P_\gamma(A) = P(A|\tau = \gamma)$$

  (regular conditional probability, given the exact outcome γ of the random object $\tau : (\Omega, \mathcal{A}) \to (\Gamma, \mathcal{S})$)
- *Interpretation*: when time starts, Nature selects which component P_γ will be active, with probability P^* on $(\Gamma, \mathcal{S})$,
  - $\Rightarrow$ the output from a stationary system always "looks ergodic," however we do not know beforehand which ergodic component will be active