Probability and Random Processes

Lecture 9

- Extensions to measures
- Product measure

Cartesian Product

- For a finite number of sets $A_1, \ldots, A_n$

$$\times_{k=1}^{n} A_k = \{(a_1, \ldots, a_n) : a_k \in A_k, k = 1, \ldots, n\}$$

  - notation A^n if $A_1 = \cdots = A_n$
- For an arbitrarily indexed collection of sets $\{A_t\}_{t \in T}$

$$\times_{t \in T} A_t = \{\text{functions } f \text{ from } T \text{ to } \cup_{t \in T} A_t : f(t) \in A_t, t \in T\}$$

  - $A_t = A$ for all $t \in T$, then $A^T = \{\text{ all functions from } T \text{ to } A\,\}$
- For a finite T the two definitions are equivalent (why?)

- For a set Ω, a collection $\mathcal{C}$ of subsets is a semialgebra if
  - $A, B \in \mathcal{C} \Rightarrow A \cap B \in \mathcal{C}$
  - if $C \in \mathcal{C}$ then there is a pairwise disjoint and finite sequence of sets in $\mathcal{C}$ whose union is C^c
- If $\mathcal{C}_1, \dots, \mathcal{C}_n$ are semialgebras on $\Omega_1, \dots, \Omega_n$ then

$$\{\times_{k=1}^n C_k : C_k \in \mathcal{C}_k,\ 1 \le k \le n\}$$

  is a semialgebra on $\times_{k=1}^n \Omega_k$

Extension

This is how we constructed the Lebesgue measure on $\mathbb{R}$:

- For any $A \subset \mathbb{R}$

$$\lambda^*(A) = \inf\left\{\sum_n \ell(I_n) : \{I_n\} \text{ open intervals, } \bigcup_n I_n \supset A\right\}$$

  (where $\ell =$ "length of interval")
- A set $E \subset \mathbb{R}$ is Lebesgue measurable if for any $W \subset \mathbb{R}$

$$\lambda^*(W) = \lambda^*(W \cap E) + \lambda^*(W \cap E^c)$$

- The Lebesgue measurable sets $\mathcal{L}$ form a σ-algebra containing all intervals
- $\lambda = \lambda^*$ restricted to $\mathcal{L}$ is a measure on $\mathcal{L}$, and $\lambda(I) = \ell(I)$ for intervals

- We started with a set function ℓ for intervals $I \subset \mathbb{R}$
  - the intervals form a semialgebra
- Then we extended ℓ to work for any set $A \subset \mathbb{R}$
  - here we used *outer measure* for the extension
- We found a σ-algebra of measurable sets,
  - based on a criterion relating to the union of disjoint sets
- Finally we restricted the extension to the σ-algebra $\mathcal{L}$, to arrive at a measure space $(\mathbb{R}, \mathcal{L}, \lambda)$

- Given Ω and and a semialgebra $\mathcal{C}$ of subsets, assume we can find a set function m on sets from $\mathcal{C}$, such that
  1. if $\emptyset \in \mathcal{C}$ (i.e. $\mathcal{C} \neq \{\Omega\}$) then $m(\emptyset) = 0$
  2. if $\{C_k\}_{k=1}^n$ is a finite sequence of pairwise disjoint sets from $\mathcal{C}$ such that $\cup_k C_k \subset \mathcal{C}$, then
  $$m\left(\bigcup_{k=1}^{n} C_k\right) = \sum_{k=1}^{n} m(C_k)$$
  3. if $C, C_1, C_2, \dots$ are in $\mathcal{C}$ and $C \subset \cup_n C_n$, then
  $$m(C) \leq \sum_n m(C_n)$$

  Call such a function m a pre-measure

- For a set Ω, a semialgebra $\mathcal{C}$ and a pre-measure m, define the set function μ^* by

$$\mu^*(A) = \inf \left\{ \sum_n m(C_n) : \{C_n\}_n \subset \mathcal{C}, \bigcup_n C_n \supset A \right\}$$

  Then μ^* is called the outer measure induced by m and $\mathcal{C}$
- A set $E \subset \Omega$ is μ^*-measurable if

$$\mu^*(W) = \mu^*(W \cap E) + \mu^*(W \cap E^c)$$

  for all $W \in \Omega$. Let $\mathcal{A}$ denote the class of μ^*-measurable sets
- $\mathcal{A} \supset \mathcal{C}$ and $\mathcal{A}$ is a σ-algebra
- $\mu = \mu^*_{|\mathcal{A}}$ is a measure on $\mathcal{A}$

The Extension Theorem

1. Given a set Ω, a semialgebra $\mathcal{C}$ of subsets and a pre-measure m on $\mathcal{C}$. Let μ^* be the outer measure induced by m and $\mathcal{C}$ and $\mathcal{A}$ the corresponding collection of μ^*-measurable sets, then
   - $\mathcal{A} \supset \mathcal{C}$ and $\mathcal{A}$ is a σ-algebra
   - $\mu = \mu^*_{|\mathcal{A}}$ is a measure on $\mathcal{A}$
   - $\mu_{|\mathcal{C}} = m$

   Also, the resulting measure space $(\Omega, \mathcal{A}, \mu)$ is complete
2. Let $\mathcal{E} = \sigma(\mathcal{C}) \subset \mathcal{A}$. If there exists a sequence of sets $\{C_n\}$ in $\mathcal{C}$ such that
   - $\cup_n C_n = \Omega$, and
   - $m(C_n) < \infty$

   then the extension $\mu^*_{|\mathcal{E}}$ is unique,
   - that is, if ν is another measure on $\mathcal{E}$ such that $\nu(C) = \mu^*_{|\mathcal{E}}(C)$ for all $C \in \mathcal{C}$ then $\nu = \mu^*_{|\mathcal{E}}$ also on $\mathcal{E}$

- Note that $\mathcal{E} \subset \mathcal{A}$ in general, and $\mu^*_{|\mathcal{E}}$ may not be complete
- In fact, $(\Omega, \mathcal{A}, \mu^*_{|\mathcal{A}})$ is the completion of $(\Omega, \mathcal{E}, \mu^*_{|\mathcal{E}})$,
  - on $\mathbb{R}$, $\mu^*_{|\mathcal{A}}$ corresponds to Lebesgue measure and $\mu^*_{|\mathcal{E}}$ to Borel measure
- Also compare the condition in 2. to the definition of σ-finite measure:
  - Given $(\Omega, \mathcal{A})$ a measure μ is σ-finite if there is a sequence $\{A_i\}$, $A_i \in \mathcal{A}$, such that $\cup_i A_i = \Omega$ and $\mu(A_i) < \infty$
- Given a space $(\Omega, \mathcal{A}, \mu)$ and its completion $(\Omega, \bar{\mathcal{A}}, \bar{\mu})$, we have

$$\bar{\mu}(B) = \inf\{\mu(A) : B \subset A \in \mathcal{A}\}$$

  for $B \in \bar{\mathcal{A}}$, and $\bar{\mu}$ is unique if μ is σ-finite
- If the condition in 2. is fulfilled for m, then $\mu^*_{|\mathcal{E}}$ is the *unique σ-finite* measure on $\mathcal{E}$ that extends m
- If the condition in 2. is fulfilled for m, then $\mu^*_{|\mathcal{A}}$ is the *unique complete and σ-finite* measure on $\mathcal{A}$ that extends m

Extension in Standard Spaces

- Consider a (metrizable) topological space Ω and assume that $\mathcal{C}$ is a algebra of subsets (i.e., also a semialgebra)
  - Algebra: closed under set complement and finite unions
- An algebra $\mathcal{C}$ has the countable extension property [Gray], if for every function m on $\mathcal{C}$ such that $m(\Omega) = 1$ and
  - for any finite sequence $\{C_k\}_{k=1}^n$ of pairwise disjoint sets from $\mathcal{C}$ we get

$$m\left(\bigcup_{k=1}^n C_k\right) = \sum_{k=1}^n m(C_k)$$

  then also the following holds:
  - If there is a sequence $\{G_n\}$, $G_n \in \mathcal{C}$, such that $G_{n+1} \subset G_n$ and $\lim \cap_n G_n = \emptyset$, then $\lim_n m(G_n) = 0$
- If $\mathcal{C}$ is (already) a σ-algebra, then these two facts (finite additivity and continuity) imply countable additivity

- Any algebra on Ω is said to be standard (according to Gray) if it has the countable extension property
- A measurable space $(\Omega, \mathcal{A})$ is standard if $\mathcal{A} = \sigma(\mathcal{C})$ for a standard $\mathcal{C}$ on Ω
- If $\mathcal{E} = (\Omega, \mathcal{T})$ is Polish, then $(\Omega, \sigma(\mathcal{E}))$ is standard
- Note that if $\mathcal{E} = (\Omega, \mathcal{T})$ is Polish, then $(\Omega, \sigma(\mathcal{E}))$ is also "standard Borel" $\Rightarrow$ for Polish spaces the two definitions of "standard" are essentially equivalent
  - again, we take the $(\Omega, \sigma(\mathcal{E}))$ from Polish space as our default standard space

Extension and Completion in Standard Spaces

- For $(\Omega, \mathcal{T})$ Polish and $(\Omega, \mathcal{A})$ the corresponding standard (Borel) space, there is always an algebra $\mathcal{C}$ on Ω with the countable extension property, and such that $\mathcal{A} = \sigma(\mathcal{C})$
- Thus, for any normalized and finitely additive m on $\mathcal{C}$
  1. m can always be extended to a measure on $(\Omega, \mathcal{A})$
  2. the extension is unique
- Let $(\Omega, \mathcal{A}, \rho)$ be the corresponding extension ($\rho(\Omega) = 1$)
- Also let $(\Omega, \bar{\mathcal{A}}, \bar{\rho})$ be the completion. Then $(\Omega, \bar{\mathcal{A}}, \bar{\rho})$ is *isomorphic mod 0* to $([0, 1], \mathcal{L}([0, 1]), \lambda)$

Product Measure Spaces

- For an arbitrary (possibly infinite/uncountable) set T, let $(\Omega_t, \mathcal{A}_t)$ be measurable spaces indexed by $t \in T$
- A measurable rectangle = any set $O \subset \times_{t \in T} \Omega_t$ of the form

$$O = \{f \in \times_{t \in T} \Omega_t : f(t) \in A_t \text{ for all } t \in S\}$$

  where S is a finite subset $S \subset T$ and $A_t \in \mathcal{A}_t$ for all $t \in S$
- Given T and $(\Omega_t, \mathcal{A}_t)$, $t \in T$, the smallest σ-algebra containing all measurable rectangles is called the resulting product σ-algebra
  - Example: $T = \mathbb{N}$, $\Omega_t = \mathbb{R}$, $\mathcal{A}_t = \mathcal{B}$ give the infinite-dimensional Borel space $(\mathbb{R}^\infty, \mathcal{B}^\infty)$

- For a finite set I, of size n, assume that $(\Omega_i, \mathcal{A}_i, \mu_i)$ are measure spaces indexed by $i \in I$
- Let $\mathcal{U} = \{$ all measurable rectangles $\}$ corresponding to $(\Omega_i, \mathcal{A}_i)$, $i \in I$
- Let $\Omega = \times_i \Omega_i$ and $\mathcal{A} = \sigma(\mathcal{U})$
- Define the product pre-measure m by

$$m(A) = \prod_i \mu_i(A_i)$$

  for any $A_i \in \mathcal{A}_i$, $i \in I$, and $A = \times_i A_i \in \mathcal{U}$

- The measurable rectangles $\mathcal{U}$ form a semialgebra
- The product pre-measure m is a pre-measure on $\mathcal{U}$

1. Given $(\Omega_i, \mathcal{A}_i, \mu_i)$, $i = 1, \dots, n$, let m be the corresponding product pre-measure. Then m can be extended from $\mathcal{U}$ to a σ-algebra containing $\mathcal{A} = \sigma(\mathcal{U})$. The resulting measure m^* is complete.
2. If each of the $(\Omega_i, \mathcal{A}_i, \mu_i)$'s is σ-finite then the restriction $m^*_{|\mathcal{A}}$ is unique.
   - Proof: $(\Omega_i, \mathcal{A}_i, \mu_i)$ σ-finite $\Rightarrow$ condition 2. on slide 8. fulfilled

- If the $(\Omega_i, \mathcal{A}_i, \mu_i)$'s are σ-finite, then the unique measure $\mu = m^*_{|\mathcal{A}}$ on $(\Omega, \mathcal{A})$ is called product measure and $(\Omega, \mathcal{A}, \mu)$ is the product measure space corresponding to $(\Omega_i, \mathcal{A}_i, \mu_i)$, $i = 1, \dots, n$

n-dimensional Lebesgue Measure

- Let $(\Omega_i, \mathcal{A}_i, \mu_i) = (\mathbb{R}, \mathcal{L}, \lambda)$ (Lebesgue measure on $\mathbb{R}$) for $i = 1, \dots, n$. Note that $(\mathbb{R}, \mathcal{L}, \lambda)$ is σ-finite. Let μ denote the corresponding product measure on $\mathbb{R}^n$
  - Per definition, the 'n-dimensional Lebesgue measure' μ constructed like this, based on 2. (on slide 8), is unique but not complete
  - Using instead the construction in 1. as the definition, we get a unique and complete version corresponding to the completion of μ
- The completion $\bar{\mu}$ of the n-product of Lebesgue measure is called n-dimensional Lebesgue measure