# Quantum Information Theory
## Spring semester, 2017

### Assignment 6
Assigned: Friday, April 28, 2017
Due: Friday, May 5, 2017 M. Skoglund

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**Problem 6.1:** Define the *quantum entropy* of a state $\rho$ and the (quantum) *mutual information* between two states $\rho_A$ and $\rho_B$

**Problem 6.2:** For two states $\rho$ and $\sigma$, define the *trace distance* $V(\rho, \sigma)$ and their *fidelity* $F(\rho, \sigma)$. Also relate the possible values for $V$ to the range of values for $F$

**Problem 6.3:** Derive an expression for the entropy of a classical–quantum state on the form

$$\sum p(x)|e(x)\rangle\langle e(x)| \otimes \sigma(x)$$

**Problem 6.4:** For the trace distance $V(\cdot, \cdot)$, prove that

$$V(\rho, \sigma) \leq V(\rho, \theta) + V(\theta, \sigma)$$

and that

$$V(\rho_1 \otimes \rho_2, \sigma_1 \otimes \sigma_2) \leq V(\rho_1, \sigma_2) + V(\rho_2, \sigma_2)$$

**Problem 6.5:** For the fidelity $F(\cdot, \cdot)$, prove that

$$F(\rho_1 \otimes \rho_2, \sigma_1 \otimes \sigma_2) = F(\rho_1, \sigma_1)F(\rho_2, \sigma_2)$$

**Problem 6.6:** The *coherent information* of $\rho_{AB} \in \mathcal{A} \otimes \mathcal{B}$ is defined as

$$Q(\rho_A\rangle\rho_B) = -S(\rho_{AB}|\rho_B) = S(\rho_B) - S(\rho_{AB})$$

with $\rho_B = \text{Tr}_{\mathcal{A}}\rho_{AB}$. Assume the dimensions of $\mathcal{A}$ and $\mathcal{B}$ are both $d < \infty$, and let $\{|e_i\rangle\}$ and $\{|f_i\rangle\}$ be a basis for $\mathcal{A}$ and $\mathcal{B}$, respectively. Consider the *maximally entangled* state $\rho_{\max} = |\psi\rangle\langle\psi|$ with

$$|\psi\rangle = \frac{1}{\sqrt{d}} \sum_{i=1}^{d} |e_i\rangle|f_i\rangle$$

Interpret "maximum entanglement" as "maximum confusion about sub-systems" by looking at $\text{Tr}_{\mathcal{A}}\rho_{\max}$ and/or $\text{Tr}_{\mathcal{B}}\rho_{\max}$. Interpret $Q(\rho_A\rangle\rho_B)$ as a measure of degree of entanglement by comparing its maximum possible value to its value for $\rho_{AB} = \rho_{\max}$ and $\rho_{AB} = \rho_A \otimes \rho_B$. Also compute $Q(\rho_A\rangle\rho_B)$ for the state

$$\rho_{AB} = \frac{1}{d} \sum_{i=1}^{d} |e_i\rangle\langle e_i| \otimes |f_i\rangle\langle f_i|$$

**Problem 6.7:** Prove that $S(\rho_A; \rho_B) = S(\rho_{AB} \| \rho_A \otimes \rho_B)$, that is, mutual information $S(\rho_A; \rho_B)$ measures distance between a possibly entangled state $\rho_{AB}$ where $\mathcal{A}$ and $\mathcal{B}$ are correlated and the non-entangled state $\rho_A \otimes \rho_B$, where $\mathcal{A}$ and $\mathcal{B}$ are independent.