Quantum Information Theory
Spring semester, 2017

Assignment 8
Assigned: Friday, May 12, 2017
Due: Friday, May 19, 2017 M. Skoglund

Problem 8.1: Define the Holevo information $\chi(\mathcal{N})$ of a channel $\mathcal{N}$.

Problem 8.2: State and explain the Holevo–Schumacher–Westmoreland coding theorem.

Problem 8.3: Define the coherent information $\mathcal{Q}(\mathcal{N})$ of a channel $\mathcal{N}$.

Problem 8.4: State and explain the coding theorem for quantum communication (preservation of entanglement) over a quantum channel $\mathcal{N}$.

Problem 8.5: Let $S(\rho_A;\rho_B)$ denote the mutual information of a state $\rho^{AB} \in \mathcal{A}\otimes\mathcal{B}$, and $S(\sigma_A;\sigma_B)$ the mutual information of another state σ^{AB}. Assume that $\mathcal{A}$ and $\mathcal{B}$ are both finite-dimensional, of dimensions d_A and d_B, respectively. Prove that if $V(\rho,\sigma) = 2^{-1}\mathrm{Tr}|\rho-\sigma| \le \varepsilon$ then

$$|S(\rho_A;\rho_B) - S(\sigma_A;\sigma_B)| \le 12\varepsilon \log d_A + 4h(2\varepsilon)$$

where $h(x) = -x\log x - (1-x)\log(1-x)$.

Problem 8.6: Assume $\{e_m\}$ is a basis for $\mathcal{M}$ and $\{f_m\}$ one for $\mathcal{M}'$, both of size M, and let

$$\theta = \frac{1}{M}\sum_m |e_m\rangle\langle e_m| \otimes |f_m\rangle\langle f_m|$$

Given $\{\rho_m^n\}$, $\rho_m^n \in \mathcal{A}^{\otimes n}$, $m \in \{1,\dots,M\}$, assume that the state

$$\frac{1}{M}\sum_m |e_m\rangle\langle e_m| \otimes \rho_m^n \in \mathcal{M}\otimes\mathcal{A}^{\otimes n}$$

is available at the input of n uses of a noisy channel $\mathcal{N}$, to form the state

$$\phi = \frac{1}{M}\sum_m |e_m\rangle\langle e_m| \otimes \mathcal{N}^{\otimes n}(\rho_m^n)$$

after transmission. Let σ be any state that can be produced by operating on ϕ at the output of the n channel uses. If for some R and any $\varepsilon > 0$ there is an N such that for all $n > N$, $V(\theta,\sigma) < \varepsilon$ and $n^{-1}\log M > R-\varepsilon$, then R is achievable for "common randomness generation" in the sense of sharing the state θ. It can be proved that any R achievable for classical communication over n uses of $\mathcal{N}$ is also achievable for common randomness generation. Use this fact, and the result in Problem 8.5, to prove a converse for the Holevo–Schumacher–Westmoreland theorem, i.e. that no rates above

$$C = \lim_{k\to\infty}\frac{1}{k}\chi(\mathcal{N}^{\otimes k})$$

are achievable for classical communication.

Problem 8.7: Determine the capacity for quantum (i.e. not classical) communication over the erasure channel

$$\rho \to (1-\varepsilon)\rho + \varepsilon|e\rangle\langle e|$$

where $|e\rangle$ is orthogonal to all eigenvectors of ρ (assuming that ρ has a finite number of eigenvectors).