Quantum Information Theory
Spring semester, 2017

Assignment 9
Assigned: Friday, May 19, 2017
Due: Friday, June 2, 2017 M. Skoglund

Problem 9.1: Define and explain the concept of *quantum error correcting code.*

Problem 9.2: Prove the Gilbert–Varshamov bound for classical linear $[n, k, d]$ codes.

Problem 9.3: Prove that a code $\mathcal{C}$, with decoder $\mathcal{D}$, that fulfills the error-correction conditions $P_{\mathcal{C}} E_i^* E_i P_{\mathcal{C}} = \gamma_{ij} P_{\mathcal{C}}$ is error correcting, i.e. $\mathcal{D}(\mathcal{E}(\rho)) = \gamma\rho$, $\gamma \in \mathbb{C}$, where $\mathcal{E}$ is the channel mapping.

Problem 9.4: [10.7 in NC] Let $\mathcal{H}$ be a qubit space with basis $\{|0\rangle, |1\rangle\}$ and consider the code $\mathcal{C} \subset \mathcal{H}^{\otimes 3}$ spanned by

$$|c_0\rangle = |0\rangle|0\rangle|0\rangle = |000\rangle, \quad |c_1\rangle = |1\rangle|1\rangle|1\rangle = |111\rangle$$

that is

$$P_{\mathcal{C}} = |000\rangle\langle 000| + |111\rangle\langle 111|$$

Let X_i be a bit-flip on the ith bit, i.e., for example $X_2|000\rangle = |010\rangle$ and assume that bit-flips happen independently with probability ε. Prove that the code is error correcting for

$$E_0 = \sqrt{(1-\varepsilon)^3} I, \;\; E_1 = \sqrt{\varepsilon(1-\varepsilon)^2} X_1, \;\; E_2 = \sqrt{\varepsilon(1-\varepsilon)^2} X_2, \;\; E_3 = \sqrt{\varepsilon(1-\varepsilon)^2} X_3$$

Problem 9.5: [10.8 in NC] For $\mathcal{H}$ with basis $\{|0\rangle, |1\rangle\}$, let $|+\rangle = 2^{-1/2}(|0\rangle + |1\rangle)$ and $|-\rangle = 2^{-1/2}(|0\rangle - |1\rangle)$. Consider the code spanned by $|c_0\rangle = |{+}{+}{+}\rangle$, $|c_1\rangle = |{-}{-}{-}\rangle$. Prove that this code can correct a single phase-flip in any of the qubits. I.e., an operation of the form

$$\alpha|0\rangle + \beta|1\rangle \to \alpha|0\rangle - \beta|1\rangle$$

in (at most) one of the qubits.

Problem 9.6: Prove that if the error-correction conditions are fulfilled for a channel with operation elements $\{E_i\}$ then they are also fulfilled for a channel where the elements are linear combinations of the E_i's.

Problem 9.7: [10.9 in NC] Use the result in Problem 9.6 to prove that the code in Problem 9.5 also corrects errors of the type

$$\alpha|0\rangle + \beta|1\rangle \to \alpha|0\rangle, \quad \alpha|0\rangle + \beta|1\rangle \to \beta|1\rangle$$

That is, projections on the basis $\{|0\rangle, |1\rangle\}$.