Quantum

Lecture 1

- Metric and linear spaces
- Hilbert spaces

The mathematical description of a quantum mechanical system relies fundamentally on Hilbert space theory

The state of a quantum system is a point in a Hilbert space $\mathcal{H}$

- A Hilbert space is a linear vector space with an inner product, which is complete relative to the induced norm

Usually it is assumed that $\mathcal{H}$ is separable, so that it has a countable basis

Performing measurements of real-valued physical entities is modeled via linear operators on $\mathcal{H}$, with the eigenvalues of the operator as possible outcomes

- To make physical sense, these operators must be self-adjoint and compact

$\Rightarrow$ we need to learn more about concepts like: *linear space, inner product, norm, complete, separable, basis, eigenvalue, self-adjoint and compact*...

Metric, Linear and Normed Spaces

For a given set Ω, a function $\rho : \Omega \times \Omega \to \mathbb{R}$ is a metric if for all $x, y, z \in \Omega$

$\rho(x, y) \geq 0$ with $=$ only if $x = y$

$\rho(x, y) = \rho(y, x)$

$\rho(x, z) \leq \rho(x, y) + \rho(y, z)$

The pair (Ω, ρ) is a metric space

For a sequence $\{x_n\}$ in a metric space Ω, if for each $\varepsilon > 0$ there is an N and an $x \in \Omega$ such that

$\rho(x_n, x) < \varepsilon$ for all $n > N$, then $\{x_n\}$ converges to the point x

$\rho(x_m, x_n) < \varepsilon$ for all $m, n > N$, then $\{x_n\}$ is a Cauchy sequence

A metric space is complete if all Cauchy sequences converge to points in the space

A set O in Ω is open if for any $x \in O$ there is an $r > 0$ such that $B_r(x) \subset O$, where $B_r(x) = \{y \in \Omega : \rho(x,y) < r\}$

Given a set $E \subset \Omega$, a point $x \in \Omega$ is a limit point of E if $O \cap E \neq \emptyset$ for all open O with $x \in O$

The set of all limit points of E = the closure of E, notation $\overline{E}$

A set E is closed if E^c is open

A set E is dense in Ω if $\overline{E} = \Omega$

- c.f. the rational numbers $\mathbb{Q}$ are dense in $\mathbb{R}$

(Ω, ρ) is separable if there is a countable set $E \subset \Omega$ such that $\overline{E} = \Omega$

- c.f. $\mathbb{R}$ is separable since $\mathbb{Q}$ is countable and $\mathbb{R} = \overline{\mathbb{Q}}$

Given a set E, a collection $\mathcal{S}$ of sets is a covering of E if $E \subset \bigcup_{S \in \mathcal{S}} S$

Given E, if $\mathcal{S}$ is a covering of E and $\mathcal{S}' \subset \mathcal{S}$ is also a covering, then $\mathcal{S}'$ is a subcovering

A covering $\mathcal{S}$ is open if it contains only open sets

Given $(\Omega, \mathcal{T})$, a subset $E \subset \Omega$ is compact if every open covering of E has a finite subcovering

- $E \subset \mathbb{R}^n$ is compact $\iff$ E is closed and bounded

(Ω, ρ) is compact if Ω is compact

A linear space is a set Ω, a field F ($= \mathbb{R}$ or $\mathbb{C}$) and two operations '$+$' and '$\cdot$' such that for all $x, y, z \in \Omega$ and $a \in F$

$x + y \in \Omega$

$a \cdot x \in \Omega$

$x + y = y + x$

$x + (y + z) = (x + y) + z$

Furthermore,

there is an element $0 \in \Omega$ such that $x + 0 = x$

there is an element $-x \in \Omega$ such that $x + (-x) = 0$

The field F is the set of scalars

A linear space is a normed space if $F = \mathbb{R}$ or $\mathbb{C}$ and there is a function $\|\cdot\|$ from Ω to $\mathbb{R}$ such that for all $x, y \in \Omega$ and $a \in F$

$\|x\| \geq 0$ with equality only if $x = 0$

$\|a \cdot x\| = |a|\|x\|$

$\|x + y\| \leq \|x\| + \|y\|$

The metric space induced by $(\Omega, \|\cdot\|)$ is the space (Ω, ρ) with $\rho(x, y) = \|x - y\|$

A normed space with metric defined by the norm is a Banach space if it is complete

Let Ω and Λ be linear with the same F. A function $L : \Omega \to \Lambda$ is a linear operator (mapping, transformation) if for $x, y \in \Omega$ and $a \in F$

$$L(x + y) = L(x) + L(y)$$
$$L(ax) = aL(x)$$

If $\sup\{\|L(x)\| : \|x\| \leq 1\} < \infty$ then L is bounded

In case $\Lambda = F$ we call L a (linear) functional

The set of all bounded L is denoted $B(\Omega, \Lambda)$. If we define

$$(L_1 + L_2)(x) = L_1(x) + L_2(x)$$

and $(aL)(x) = aL(x)$ for $L, L_1, L_2 \in B(\Omega, \Lambda)$ and any $a \in F$, then $B(\Omega, \Lambda)$ is a linear space, and a normed space with norm

$$\|L\| = \sup\{\|L(x)\| : \|x\| \leq 1\} < \infty$$

When $\Lambda = F$ we say $B(\Omega, \Lambda) = \Omega^*$ with norm $\|\cdot\|_*$

For a normed space Ω, the space $(\Omega^*, \|\cdot\|_*)$ of bounded linear functionals on Ω is called the dual of Ω

Let Ω be a linear space with scalar field $\mathbb{R}$ or $\mathbb{C}$. An inner product on Ω is a mapping $g : \Omega \times \Omega \to F$ such that for all $x, y, z \in \Omega$ and scalars a and b, and with $\langle x, y \rangle = g(x, y)$

$\langle ax + by, z \rangle = a\langle x, z \rangle + b\langle y, z \rangle$

$\langle x, y \rangle = (\langle y, x \rangle)^*$ (complex conjugate)

$\langle x, x \rangle \geq 0$ with equality only if $x = 0$

The pair (Ω, g) is an inner product space

Given (Ω, g), with $\langle x, y \rangle = g(x, y)$, $\|x\| = \sqrt{\langle x, x \rangle}$ is a norm

For two inner product spaces (Ω, g) and (Γ, u), a linear operator $L : \Omega \to \Gamma$ is unitary if it is bijective (1-to-1 and onto) and $u(L(x), L(y)) = g(x, y)$ for all $x, y \in \Omega$

Hilbert Space

A complete normed inner product space with norm induced by the inner product is called a Hilbert space

x and y (in a Hilbert space) are orthogonal if $\langle x, y \rangle = 0$

For $S \subset \Omega$, the orthogonal complement is

$$S^{\perp} = \{y \in \Omega : \langle x, y \rangle = 0 \text{ for all } x \in S\}$$

Let (Ω, g) be an inner product space, then there exists a Hilbert space $(\mathcal{H}, h)$, the completion of (Ω, g), fulfilling:

There is a linear 1-to-1 mapping $T : \Omega \to \mathcal{H}$ such that $g(x, y) = f(T(x), T(y))$

$\{T(x) : x \in \Omega\}$ is dense in $\mathcal{H}$

$S \subset \Omega$ is an orthogonal set if all elements are pairwise orthogonal; such a set is orthonormal if in addition $\|x\| = 1$ for all $x \in S$

If S is orthonormal and is not contained in any strictly larger orthonormal set, then S is a basis

If $\Omega \neq \{0\}$ is a Hilbert space, then it has a basis S

In general S can be uncountable; a Hilbert space has a countable basis iff it is separable

- assume that whenever we have a Hilbert space it is separable

The span, span(S), of an arbitrary subset S is the set of all finite linear combinations of the elements in S

For a (separable) Hilbert space Ω and an orthonormal subset $E \subset \Omega$, the following are equivalent

E is a basis

$\overline{\text{span}(E)} = \Omega$

$\langle x, e \rangle = 0$ for each $e \in E \Rightarrow x = 0$

for each $x \in \Omega$

$$x = \sum_{e \in E} \langle x, e \rangle e \quad \text{and} \quad \|x\|^2 = \sum_{e \in E} |\langle x, e \rangle|^2$$

For Ω and its dual Ω^*, $\ell \in \Omega^*$ iff there is a $y \in \Omega$ such that $\ell(x) = \langle y, x \rangle$ for each $x \in \Omega$. Also, $\|\ell\|_* = \|y\|$

Compact Self-Adjoint Operators

Let $\Omega \neq \{0\}$ be a Hilbert space and $T : \Omega \to \Omega$ a bounded linear operator. For fixed y the mapping $x \to \langle T(x), y \rangle$ is a bounded linear functional

There is a unique $t \in \Omega$ such that $\langle T(x), y \rangle = \langle x, t \rangle$

The element t is called the adjoint of T, notation $T^*(y)$

If $T^*(x) = T(x)$ for all $x \in \Omega$ then T is self-adjoint

A linear operator T is compact if $\overline{\{T(x) : \|x\| \leq 1\}}$ is compact

A number λ is an eigenvalue of T if there is a $x \neq 0$ such that $T(x) = \lambda x$, where x is the corresponding eigenvector

If T is self-adjoint, then

- $\langle T(x), x \rangle$ is real-valued for all $x \in \Omega$
- all eigenvalues are real
- if $\lambda_1 \neq \lambda_2$ correspond to x_1 and x_2, then $\langle x_1, x_2 \rangle = 0$
- if λ is an eigenvalue then $|\lambda| \leq \|T\|$
- if T is also compact then for $\lambda \neq 0$ the set $\{x : T(x) = \lambda x\}$ is finite-dimensional (has a finite basis)

Spectral Theorem. Let T be a compact self-adjoint operator on Ω. Then there is a countable orthonormal set $\{u_n\}$ such that for any $x \in \Omega$

$$T(x) = \sum_n \lambda_n \langle x, u_n \rangle u_n$$

where $\{\lambda_n\}$ are all non-zero (not necessarily distinct) eigenvalues

Note that we can write

$$T(x) = \sum_i \lambda_i P_i(x)$$

over *distinct* eigenvalues λ_i, where $P_i(x)$ projects x to $\{x : T(x) = \lambda_i x\}$, the finite-dimensional subspace spanned by the eigenvectors of λ_i

If $\Omega \neq \{0\}$ is a separable Hilbert space and T a compact self-adjoint operator, then the eigenvectors of T form a countable basis for Ω

For a bounded linear operator T on a Hilbert space Ω, the trace of T is defined as

$$\operatorname{Tr} T = \sum_i \langle e_i, T(e_i) \rangle$$

where $\{e_i\}$ is any orthonormal basis. The number $\operatorname{Tr} T$ does not depend on which basis we choose

An operator T is positive if it is self-adjoint and $\langle x, T(x) \rangle > 0$ for all $x \Rightarrow \operatorname{Tr} T > 0$

For any positive operator T, let $\sqrt{T}$ be the operator that solves $\sqrt{T}(\sqrt{T}(x)) = T(x)$

If $T : \Omega_1 \to \Omega_2$ is bounded then $T^*(T(x)) : \Omega_1 \to \Omega_1$ is positive

Let $|T|(x) = \sqrt{T^*(T(x))}$ be the absolute value of T

A bounded operator T is trace class if $\operatorname{Tr}|T| < \infty$

trace class $\Rightarrow$ compact

The exponential of a bounded linear operator

$$\exp T = I + T + \frac{T^2}{2} + \frac{T^3}{3!} + \cdots$$

(where I is the identity, $I(x) = x$, and $T^2(x) = T(T(x))$, etc.)

The bounded linear operator T has a logarithm L, if $T = \exp L$